Moduli spaces of curves and moduli spaces of cubic surfaces

Discussed with Professor Benson Farb

1 Introduction

We construct moduli spaces to capture how structures vary within families. The global topological and geometric properties of these spaces reflect the properties of the objects they parametrize. Understanding moduli spaces provides insights into these objects. In this topic proposal we present two examples of moduli spaces that illustrate this philosophy.

An important example is the moduli space of Riemann surfaces $\mathcal{M}_g$, which parametrizes complex structures on a topological genus g surface S_g up to biholomorphisms. Equivalently, this classifies hyperbolic structures on S_g up to isometries. Examining the geometry of the moduli spaces tells us about the behavior of hyperbolic structures on surfaces under deformation. Just as one studies an orbifold by lifting to its universal cover we can study $\mathcal{M}_g$ by considering the *Teichmüller space* $\mathrm{Teich}(S_g)$, which consists of marked hyperbolic structures on S up to isotopy. The *mapping class group* $\mathrm{Mod}(S_g)$ plays a key role here, acting properly discontinuously on $\mathrm{Teich}(S_g)$ by change of marking. The quotient $\mathrm{Teich}(S_g)/\mathrm{Mod}(S_g)$ is precisely the moduli space $\mathcal{M}_g$.

Section 2 covers the basic theory of mapping class groups of surfaces, including first examples, generators and relations, and some important subgroups. In Section 3, we will discuss the topology and geometry of Teichmüller spaces and its relation to the moduli spaces of Riemann surfaces.

In Section 4, we will switch our attention to another type of moduli spaces that naturally show up in enumerative geometry. We will construct the moduli space of smooth cubic surfaces in $\mathbb{CP}^3$ and show that there are exactly 27 lines on each one of them leveraging the covering space structure of an incidence correspondence. Furthermore, we will discuss the automorphisms of the collection of 27 lines and its connections to the moduli space of cubic surfaces.

2 Mapping Class Groups of Surfaces

The classification of surfaces shows that the homeomorphism types of finite-type surfaces are completely determined by their genus g, number of boundary components b, and number of punctures n. We denote such a surface by $S_{g,n}^b$ in this proposal and omit the parameters that are 0. While a topological classfication is fully realized, there is no canonical homeomorphism between two surfaces of the same type. Mapping class groups naturally arise as we seek to understand the distinct ways a surface can be homeomorphic to another. These groups have broad applications in higher dimensions, such as describing how three-manifolds can be constructed by gluing along boundary surfaces and studying surface bundles through their monodromy representations.

Definition 2.1. *The mapping class group of a surface S possibly with boundary ∂S, denoted by* $\mathrm{Mod}(S)$*, is the group*

$$\mathrm{Mod}(S) = \mathrm{Homeo}^{+}(S, \partial S) / \mathrm{Homeo}_0(S, \partial S) \cong \pi_0(\mathrm{Homeo}^{+}(S, \partial S)).$$

That is the group of oriention-preserving homeomorphisms of the surface S modulo the relation of isotopy. In the presence of punctures, we define the pure mapping class group of S to be the subgroup *of* $\mathrm{Mod}(S)$ *that fixes all punctures pointwise, denoted by* $\mathrm{PMod}(S)$.

In $\mathrm{Mod}(S)$, there is a particular type of mappping class called *Dehn twists.* Cutting out a tubular neighborhood of a nonseparating curve α in S, we can twist one boundary component of the annulus by 2π and glue it back to the reset of the surface. The following result states that such mapping classes serve as generators for pure mapping class groups.

Theorem 2.1 (Finite Generation, [FM12, Theorem 4.1])**.** *For $g \geq 1$,* $\mathrm{PMod}(S_{g,n})$ *is generated by finitely many Dehn twists about nonseparating simple closed curves.*

Proof Sketch. The proof proceeds by first inducting on the number of punctures n and then on the genus g.

For induction on n, we begin with the base case of the torus T^2, where $\mathrm{Mod}(T) \cong \mathrm{SL}(2; \mathbb{Z})$ is indeed generated by two Dehn twists. Then we use the Birman short exact sequence

$$1 \to \pi_1(S_{g,n}) \xrightarrow{Push} \mathrm{PMod}(S_{g,n+1}) \to \mathrm{PMod}(S_{g,n}) \to 1$$

for the inductive step, where $\mathrm{PMod}(S_{g,n})$ is finitely generated by Dehn twists by induction hypothesis. Since the image of $\pi_1(S_{g,n})$ under the Push map is finitely generated by Dehn twists, this finishes the induction on n.

For induction on g, we assume that the pure mapping class group $\mathrm{PMod}(S_{g-1,n})$ is finitely generated by Dehn twists for all n. We construct the *curve complex* $\widehat{\mathcal{N}}(S_{g,n})$ where vertices represent isotopy classes of simple closed curves, and edges connect two vertices if their isotopy classes have geometric intersection number exactly 1. The key step here is to prove the connectedness of the $\widehat{\mathcal{N}}(S_{g,n})$ for $g \geq 2$ through a detaild analysis of curves on $S_{g,n}$. Using the connectedness, we can assume that our mapping class fixes an isotopy class of simple closed curve a, up to composing with finitely many Dehn twists. Cutting the surface along a, we can now apply the induction hypothesis to the resulting surface $\mathrm{PMod}(S_{g-1,n+2})$. □

Explicit relations among Dehn twists have been studied, and in some cases, finite presentations have been established [FM12]. Here, we highlight some of the basic relations involving two Dehn twists.

Proposition 2.1 ([FM12, Chapter 3])**.** *Let a and b be two isotopy classes of simple closed curves on a surface S. Let T_a and T_b denote the Dehn twists about two simple closed curves representing a and b respectively. Then the relations between T_a and T_b depend on the geometric intersection number $i(a, b)$ in the following way:*

1. *(commuting relation) If $i(a, b) = 0$, then $T_a T_b = T_b T_a$.*

2. *(braid relation) If* $i(a,b) = 1$*, then* $T_aT_bT_a = T_bT_aT_b$.

3. *(no relation) If* $i(a,b) \geq 2$*, then* T_a, T_b *generate a rank-2 free subgroup of the mapping class group.*

Aside from the binary relations listed above, there are relations involving three or more Dehn twists. For example, we have the lantern relation as illustrated in Figure 1, given by Dehn twists about curves on S_0^4.

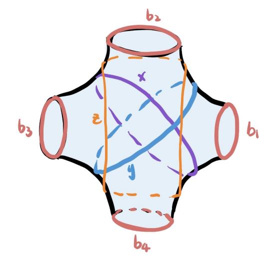

Figure 1: lantern relation: $T_xT_yT_z = T_{b_1}T_{b_2}T_{b_3}T_{b_4}$

Algebraic propeties of the mapping class groups can be deduced from these relations. Using the same example as above, we can deduce from the lantern relation that the first homology of mapping class groups for mapping class groups of S_g for $g \geq 3$ must be trivial, i.e., $H_1(\mathrm{Mod}(S_g);\mathbb{Z}) = 1$ for $g \geq 3$ [FM12].

2.1 Subgroups of Mapping Class Groups

In this subsection, we describe two types of subgroups of mapping class groups: finite subgroups and torsion free subgroups of finite index.

To understand finite subgroups of $\mathrm{Mod}(S)$, we will leverage hyperbolic geometry via the Nielsen Realization theorem. Recall that symmetries on hyperbolic surfaces are described by isometries. For a closed hyperbolic surface, there are finitely many isometries. The following theorem provides an explicit upper bound for the size of the group of orientation-preserving isometries.

Theorem 2.2 ($84(g-1)$ theorem, [FM12, Theorem 7.4])**.** *If* X *is a closed hyperbolic surface of genus* $g \geq 2$*, then* $|\mathrm{Isom}^+(X)| \leq 84(g-1)$.

Note that for any hyperbolic surface, the only isometry homotopic to identity is identity itself. As a consequence, isomtries coming from hyperbolic structures allow us to construct finite subgroups of $\mathrm{Mod}(S_g)$. This idea is formalized through the Nielsen Realization Theorem, which says that every finite subgroup of $\mathrm{Mod}(S_g)$ could be realized by an isometry group $\mathrm{Isom}^+(S_g)$ for $g \geq 2$. Nielsen proved the case for cyclic groups [Nie42]. Here We give the general case proved by Kerckhoff.

Theorem 2.3 (Nielsen Realization Theorem, [FM12])**.** *Let* $S = S_{g,n}$ *with* $\chi(S) < 0$. *Suppose* $G \leq \mathrm{Mod}(S)$ *is a finite group, then there exists a finite group* $\tilde{G} \leq \mathrm{Homeo}^+(S)$ *so that the natural projection* $\mathrm{Homeo}^+(S) \to \mathrm{Mod}(S)$ *restricts to an isomorphism* $\tilde{G} \to G$. *Furthermore,* $\tilde{G}$ *can be chosen to be a subgroup of isometries of some hyperbolic metric on* S.

We will give a proof of the special case of this theorem when G is cyclic in Section 3, using the topological structure of Teichmuller spaces.

Assuming this theorem, any finite subgroup of $\mathrm{Mod}(S_g)$ for $g \geq 2$ can be realized as a group of isometries of some hyperbolic surface.

Now we switch our focus to constructing torsion free subgroups of $\mathrm{Mod}(S_g)$ of finite index. The action of the mapping class group $\mathrm{Mod}(S_g)$ on the first integral homology $H_1(S_g;\mathbb{Z})$ provides a way to understand the group $\mathrm{Mod}(S_g)$ through the representation

$$\Psi : \mathrm{Mod}(S_g) \to \mathrm{Aut}(H_1(S_g;\mathbb{Z})).$$

Since the algebraic intersection number on $H_1(S_g;\mathbb{R})$ endows it with a symplectic structure which is preserved by mapping classes, we have an induced map $\mathrm{Mod}(S_g) \to \mathrm{Sp}(2g,\mathbb{Z})$, where $\mathrm{Sp}(2g,\mathbb{Z})$ is the integral symplectic group.

Let the *level m congruence subgroup* $\mathrm{Sp}(2g,\mathbb{Z})[m]$ of $\mathrm{Sp}(2g,\mathbb{Z})$ be the kernel of the reduction homomorphism:

$$\mathrm{Sp}(2g,\mathbb{Z})[m] = \ker(\mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}/m\mathbb{Z})).$$

We get the following property about this congruence subgroup using modular arithmetic.

Lemma 2.1. *[FM12, Theorem 6.8] For $g \geq 1$, the congruence subgroup $\mathrm{Sp}(2g,\mathbb{Z})[m]$ is torsion free for $m \geq 3$.*

These torsion free subgroups pull back to subgroups of $\mathrm{Mod}(S_g)$ through the symplectic representation Ψ. In specific, we define the *level m congruence subgroup* $\mathrm{Mod}(S_g)[m]$ as the kernel of the composition:

$$\mathrm{Mod}(S_g)[m] = \ker(\mathrm{Mod}(S_g) \xrightarrow{\Psi} \mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}/m\mathbb{Z})).$$

To prove that the subgroups $\mathrm{Mod}(S_g)[m]$ are torsion free in $\mathrm{Mod}(S_g)$, we also need the following lemma stating that torsion elements in $\mathrm{Mod}(S_g)$ could be observed through the symplectic representation.

Lemma 2.2. *[FM12, Theorem 6.9] Let $g \geq 1$. If a nontrivial mapping class $f \in \mathrm{Mod}(S_g)$ is of finite order, then $\Psi(f)$ is not trivial.*

Proof. The result is immediate for $g = 1$. We assume $g \geq 2$. Suppose $f \in \mathrm{Mod}(S_g)$ is of finite order. Then by Nielsen Realization, we can assume f is represented by a hyperbolic isometry with respect to some hyperbolic metric. Isometries can only have isolated fixed point of index 1, Lefschetz theorem applies and we have that Lefschetz number

$$L(\phi) = 2 - \mathrm{Tr}\colon (\phi_*\colon \mathrm{H}_1(\mathrm{S_g};\mathbb{Z}) \to \mathrm{H}_1(\mathrm{S_g};\mathbb{Z})) \geq 0.$$

So $\Psi(f) = \phi_*$ cannot be identity for $g \geq 2$. □

Theorem 2.4. *For $g \geq 1$, the congruence subgroup $\mathrm{Mod}(S_g)[m]$ is torsion free for $m \geq 3$.*

Furthermore, these congruence subgroups $\mathrm{Mod}(S_g)[m]$ have finite index in $\mathrm{Mod}(S_g)$, since $\mathrm{Sp}(2g,\mathbb{Z}/m\mathbb{Z})$ is finite.

Now we transition from discussing the group-theoretical properties of mapping class groups to exploring their relationship with the Teichmüller space $\mathrm{Teich}(S_g)$. Many of these group-theoretical properties will now be translated into topological properties of $\mathrm{Teich}(S_g)$.

3 Teichmüller spaces and moduli spaces of surfaces

In this section, we will introduce the Teichmüller space $\mathrm{Teich}(S_g)$ and its relation to $\mathrm{Mod}(S_g)$. For this section, let S be a surface with $\chi(S) < 0$.

3.1 Teichmüller Spaces

Definition 3.1. *Let S be a surface with $\chi(s) < 0$. A* marked hyperbolic structure (X, ϕ) *is a diffeomorphism $S \to X$, where X is a surface with a complete finite-area hyperbolic metric with totally geodesic boundary. The diffeomorphism ϕ is referred to as the* marking. We say two hyperbolic structures (X_1, ϕ_1) and (X_2, ϕ_2) are *homotopic* if there is an isometry $g : X_1 \to X_2$ so that $g \circ \phi_1$ is homotopic to ϕ_2. The Teichmüller space $\mathrm{Teich}(S)$ is the space of marked hyperbolic structures up to homotopy.

We can define a topology on $\mathrm{Teich}(S)$ through an algebraic approach. A point in $\mathrm{Teich}(S_g)$ for $g \geq 2$ gives a discrete faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ up to conjugation by $\mathrm{PGL}(2, \mathbb{R})$. If we consider the set of all discrete faithful representations up to conjugacy, denoted by $\mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2, \mathbb{R}))/\,\mathrm{PGL}(2, \mathbb{R})$, points in this space naturally correspond to points in $\mathrm{Teich}(S_g)$. Note that the space $\mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2, \mathbb{R}))/\,\mathrm{PGL}(2, \mathbb{R})$ carries a natural topology, inherited from the Lie group topology of $\mathrm{PSL}(2, \mathbb{R})^{2g}$ by picking where to send a set of $2g$ generators. With this topology, if we take any isotopy class of simple closed curve α on S_g, then the length function $l(\alpha) : \mathrm{Teich}(S_g) \to \mathbb{R}; X \mapsto l_X(\alpha)$, measuring the length of the representing geodesic of the image of α, is continuous. This property allows us relate the topology of the parametrizing space $\mathrm{Teich}(S_g)$ to lengths of geodesics on specific parametrized objects .

The following theorem shows that the Teichmüller space of S_g is homeomorphic to open balls of dimension $6g - 6$.

Theorem 3.1 (Fenchel-Nielsen Coordinates, [FM12, Theorem 10.6]). *For $g \geq 2$, we have* $\mathrm{Teich}(S_g) \cong \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$.

We offer a heuristic argument for this. For each closed surface S_g for $g \geq 2$, a *pants decomposition* is a maximal collection of disjoint, non-isotopic, essential simple closed curves. A pants decomposition of S_g consists of exactly $3g - 3$ such curves, dividing S_g into $2g - 2$ pairs of pants. We call these curves the pants curves. Fixing a pants decomposition P, every marked hyperbolic structure (X, ϕ) decomposes into $2g - 2$ hyperbolic pairs of pants along the hyperbolic geodesics representing the pants curves.

Given a hyperbolic pair of pants P, for each pair of distinct boundary geodesics α_i and α_j, there is a unique embedded geodesic arc orthogonally connecting them. We call these arcs seam curves. Cutting along the seam curves yields two isometric right-angled hyperbolic hexagons. The following proposition tells us that three parameters are sufficient to parametrize right-angled hyperbolic hexagons up to hyperbolic isometries.

Lemma 3.1. *[FM12, Proposition 10.4] Given $(x, y, z) \in \mathbb{R}_+^3$, there exists a hyperbolic hexagon whose lengths of three alternating sides are given by this tuple, and this hexagon is unique up to* $\mathrm{PSL}(2, \mathbb{R})$.

If we glue up two isometric right-angled hyperbolic hexagons, we obtain a hyperbolic structure on a pair of pants. Translating Proposition 3.1 through the bijection between Teich(P) and the set of oriented isometry classes of marked right-angled hyperbolic hexagons, we can derive the following proposition describing Teich(P).

Lemma 3.2. *[FM12, Proposition 10.5] Let $\mathcal{P}$ be a pair of pants with boundary components α_1, α_2, and α_3. Then the map* $\text{Teich}(P) \to \mathbb{R}_+^3$ *defined by*

$$\chi \mapsto (l_\chi(\alpha_1), l_\chi(\alpha_2), l_\chi(\alpha_3))$$

is a homeomorphism, where $l_\chi(\alpha_i)$ is the length of the curve α_i.

Now to reconstruct the hyperbolic surface X from pairs of pants, we need to specify how the surface was glued from the $2g-2$ pairs of pants P_i. In particular, boundary circles are glued in pairs using isometries, requiring exactly $3(2g-2)/2 = 3g-3$ length parameters. In addition, we can twist one of the boundary circles by any angle $\theta \in \mathbb{R}$ before glueing onto the other one, resulting in different marked hyperbolic structures. For example, if we twist one boundary circle by exactly 2π, then this corresponds to a Dehn twist of the initial marking about the boundary circle. Thus, we need exactly $3g-3$ twist parameters, along with $3g-3$ length parameters, to determine a hyperbolic structure. In fact, different twisting parameters give different hyperbolic strucutres. This concludes the discussion of Theorem 3.1. The homeomorphism $\mathbb{R}_+^{3g-3} \times \mathbb{R}^3 \cong \text{Teich}(S)$ gives a system of *Fenchel-Nielsen coordinates.*

Now we take a short detour to use this topological structure of the space Teich(S_g) as a contractible space to prove a special case of Theorem 2.3, Nielsen Realization for cyclic groups.

Proof Sketch of Nielsen Realization for Cyclic Groups. Suppose $f \in \text{Mod}(S_g)$ has finite order n. Then the cyclic group $H = \langle f \rangle$ acts on Teich(S_g). If H acts on Teich(S_g) properly discontinuously and freely, then the quotient Teich$(S_g)/H$ would be a finite-dimensional model of a classifying space of H. However, classifying spaces of cyclic groups are infinite dimensional. So f^k must have a fixed point for some $k \in \mathbb{N}$. If n is prime, this implies that f has a fixed point. If not, we induct on the number of prime factors of n. □

So far we discussed the topology of Teich(S_g). In fact, for any two marked hyperbolic structures (X,ϕ) and (Y,ψ), or equivalently two conformal structures up to equivalence, there is a natural way to compare them and thus gives rise to a notion of a distance. In specific, Teichmüller proved that there exists a unique *quasiconformal homeomorphism* $X \to Y$ that minimizes the *quasiconformal dilation* in a given homotopy class [FM12, Theorem 11.8, 11.9]. If we consider the homeomorphisms $X \to Y$ that corresponds to maps homotopic to $id : S_g \to S_g$, there is a unique quasiconformal one that distorts angles the least, called the *Teichmüller map.* Let K be the dilation, then we can define the distance $d_{\text{Teich}}((X,\phi),(Y,\psi)) = \frac{1}{2}\log K$. This distance function induces a metric on Teich(S_g) called the *Teichmüller metric.* This additional structure allows us to describe how lengths of curves change from a hyperbolic surface to another given the Teichmüller distance betwen them, which is formalized by Wolpert's lemma in the next subsection.

3.2 Moduli Spaces

The mapping class group $\mathrm{Mod}(S) \cong \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S)$ acts on $\mathrm{Teich}(S)$. Let $f \in \mathrm{Mod}(S)$. Suppose f is represented by some $\phi \in \mathrm{Diff}^+(S)$. Then we define the action by precomposition of marking. That is, $f \cdot (X, \psi) = (X, \psi \circ \phi^{-1})$. This is well-defined since isotopic maps give rise to homotopic hyerbolic structures. Futhermore, we can see from this definition that $f \cdot (X, \psi)$ is the point in $\mathrm{Teich}(S)$ given by the same hyperbolic surface, but with a different marking. We will show that this action is properly discontinuous, exploiting the following lemma of hyperbolic geometry on the level of spaces.

Lemma 3.3 (Discreteness of Raw Length Spectrum, [FM12, Lemma 12.4]). *Let X be a compact hyperbolic surface and let $L > 0$. Then the number of closed geodesics of length less than L is finite.*

Now we will link this geometric property about specific hyperbolic surfaces to Teichmüller space through the following lemma.

Lemma 3.4 (Wolpert's Lemma, [FM12, Lemma 12.5]). *Let X_1 and X_2 be hyperbilic surfaces and let $\phi : X_1 \to X_2$ be a K-quasiconformal homeomorphism. For any isotopy class c of simple closed curves in X_1, the following inequalities hold:*

$$\frac{l_{X_1}(c)}{K} \le l_{X_2}(\phi(c)) \le K l_{X_1}(c).$$

Theorem 3.2 (Fricke, [FM12]). *The action of* $\mathrm{Mod}(S)$ *on* $\mathrm{Teich}(S)$ *is properly discontinuous.*

Proof. Let $B \subseteq \mathrm{Teich}(S)$ be a compact domain. We need to show that $\{f \in \mathrm{Mod}(S) : f \cdot B \cap B \neq \varnothing\}$ is finite. Consider a finite collection S of isotopy classes of closed curves such that only the trivial mapping class fixes all of them and let L be their maximal length. Choose a point $(X, \phi) \in B$. Then any $f \in \mathrm{Mod}(S)$ such that $f \cdot B \cap B \neq \varnothing$, the new marking $f \cdot (X, \phi) \in \mathrm{Teich}(S)$ is at most distance $2 \cdot \mathrm{Diam}(B)$ away from (X, ϕ) with respect to the Teichmüller metric. By Wolpert's lemma, the lengths of curves in S with respect to $f \cdot (X, \phi)$ are bounded by $e^{4 \cdot \mathrm{Diam}(B)} L$. According to Lemma 3.3, there are only finitely many isotopy classes of curves with lengths bounded above by $e^{4 \cdot \mathrm{Diam}(B)} L$. Since there are only finitely many maps from S to the set of isotopy classes of curves of lengths at most $e^{4 \cdot \mathrm{Diam}(B)} L$, there can be at most finitely many f such that $f \cdot B \cap B \neq \varnothing$. □

Remark 3.1. *With respect to the Teichmüller metric, the action* $\mathrm{Mod}(S)$ *on* $\mathrm{Teich}(S)$ *is through isometries.*

Now let $g \geq 2$. Since $\mathrm{Mod}(S_g)$ acts on $\mathrm{Teich}(S_g) \cong \mathbb{R}^{6g-6}$ properly discontinuously by change of marking, we can take the quotient to get the moduli space of hyperbolic structures, where we forget the marking information.

Definition 3.2. *The moduli space of genus-g Riemann surfaces is*

$$\mathcal{M}(S_g) = \mathrm{Teich}(S_g)/\mathrm{Mod}(S_g).$$

Note that the mapping class group $\mathrm{Mod}(S_g)$ does not act freely on $\mathrm{Teich}(S_g)$, and the fixed points come from the symmetries of hyperbolic structures. For example, any element in $\mathrm{Mod}(S)$ that can be represented by an orientation-preserving isometry with respect to a hyperbolic structure X fixes (X, ϕ). The stabilizer of a point $(X, \phi) \in \mathrm{Teich}(S)$ is given by mapping classes of orientation-preserving isometries of the hyperbolic surface X, the size of which is upper bounded. Since the mapping class group action is not free, quotient $\mathcal{M}(S_g)$ is an orbifold rather than a manifold. However, this is the nicest type of orbifold, as it is finitely coverd by a manifold. Recall that for $m \geq 3$, the level m congruence subgroup $\mathrm{Mod}(S_g)[m] \leq \mathrm{Mod}(S_g)$ is a torsion free subgroup of finite index. Since $\mathrm{Mod}(S_g)$ acts properly discontinuously on $\mathrm{Teich}(S_g)$, the stabilizer of any point $x \in \mathrm{Teich}(S_g)$ is finite and thus not contained in $\mathrm{Mod}(S)[m]$. Hence the action of $\mathrm{Mod}(S)[m]$ on $\mathrm{Teich}(S)$ is free and the quotient $\mathcal{M}_g[m] := \mathrm{Teich}(S)/\mathrm{Mod}(S)[m]$ is a manifold. This manifold is a normal covering space of the moduli space, via the projection $\mathcal{M}_g[m] \to \mathcal{M}_g$, and the deck group given by $\mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$

The space $\mathcal{M}(S_g)$ is not compact, and this is observed by a geometric invariant of hyperbolic surfaces. Let $l(X)$ be the length of the shortest essential closed geodesic in X. Mumford's compactness criterion uses this quantity $l(X)$ to give explicit descriptions of compact subspaces, as illustrated as follows:

Theorem 3.3 (Mumford's Compactness Criterion)**.** *Let $g \geq 1$. For each $\varepsilon > 0$, the space $\mathcal{M}_\varepsilon(S_g) = \{X \in \mathcal{M}(S_g) : l(X) \geq \varepsilon\}$ is compact.*

Lemma 3.5 (Bers' Constant)**.** *Let S be a compact surface with $\chi(S) < 0$. There is a constant $L = L(S)$ such that for any hyperbolic surface X homeomorphic to S, there is a pants decomposition of X such that all the pants curves have lengths bounded above by L.*

Remark 3.2. *This constant L only depends on the underlying topological structure.*

Proof of the Mumford's Compactness Criterion. When $g = 1$, the compactnes of $\mathcal{M}_\varepsilon(S_1) = \mathcal{M}_\varepsilon(T^2)$ follows from the fact that any infinite sequence of lattices has a convergent subsequence, where the limit is non-degenerate.

For $g \geq 2$, we can show that $\mathcal{M}_\varepsilon(S_g)$ is sequentially compact for any ε. Consider a sequence in $X_i \in \mathcal{M}_\varepsilon(S_g)$ and lift all elements to $\mathcal{X}_i \in \mathrm{Teich}(S_g)$. For each lift, there eixsts a pants decomposition P_i so that all pants curves have lengths bounded above by L. Up to mapping class group action, we may assume that there is a subsequence $\mathcal{X}_{i_j}$ with $P_{i_j} = P$ for some pants decomposition P and thus all length parameters are bounded above by L by Lemma 3.5 and bounded below by ε. Twisting factors can also be adjusted by mapping classes to be in the range $[0, 2\pi]$. Thus we can find a subsequence of $X_i \in \mathcal{M}_\varepsilon(S_g)$ that lifts to a compact box in $\mathrm{Teich}(S_g)$ and thus must converge. □

Since $M(S_g) = \bigcup_{\varepsilon > 0} \mathcal{M}_\varepsilon(S_g)$, Theorem 3.3 tells us that the only way for a sequence to leave every compact set in $\mathcal{M}(S_g)$ is by pinching some simple closed curve short.

4 Cubic Surfaces

In this section, we discuss the second example of a moduli space of varieties.

4.1 Hypersurfaces in $\mathbb{CP}^n$

Degree d hypersurfaces in $\mathbb{CP}^n$ are defined as $V(f) = f^{-1}(0)$ for a degree d homogeneous polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$. This polynomial has equation $f = \sum_I a_I x^I$ where x^I represents a degree d monomial in $x_0, \dots, x_n$. There are $\binom{n+d+1}{d}$ monomials of degree d in total, and two choices of coefficients give the same hypersurface if and only if they are scalar multiples of each other. Thus we can parametrize all such surfaces by projective space $\mathbb{CP}^N := \mathbb{CP}^{\binom{n+d}{d}-1}$. A hupersurface $V(f)$ is called *singular* if there exists $[p_0 : \dots : p_n] \in \mathbb{CP}^n$ such that $\frac{\partial f}{\partial x_i}(p_0, \dots, p_n) = 0$ for all $0 \le i \le n$. Let $\Sigma_{d,n}$ denote the set of singular hypersurfaces of degree d in $\mathbb{CP}^n$, called the *singular locus* . This is an algebraic set given by a polynomial in the coefficients $\{a_I\}$, called the *resultant* . Carving out this singular locus, we have the space of smooth degree d hypersurfaces, called $X_{d,n}$. Note that the space $X_{d,n} \subseteq \mathbb{CP}^N$ is a path-connected smooth manifold because $\Sigma_{d,n}$ has real codimension 2 and is closed with respect to the analytic topology. This topological property gives rise to the following statement about degree d hypersurfaces in $\mathbb{CP}^n$.

Theorem 4.1. *Any two smooth degree d hypersurfaces in $\mathbb{CP}^n$ are diffeomorphic for $n \ge 1$ and $d \ge 1$.*

Proof. Consider the universal family of degree d hypersurface in $\mathbb{CP}^n$, defined as

$$U_{d,n} = \{(X, p) : X \in X_{d,n} p \in M\} \subset X_{d,n} \times \mathbb{CP}^n.$$

We have an incidence variety projection $U_{d,n} \xrightarrow{\pi} X_{d,n}; (M, p) \mapsto M$, which is a proper submersion. By Ehresman's theorem, this forms a fiber bundle. Since $X_{d,n}$ is path-connected, all fibers are diffeomorphic, and these fibers are precisely the smooth degree d hypersurfaces. □

For the remainder of this proposal, we will focus on cubic surfaces in $\mathbb{CP}^3$, which are by degree 3 homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_3]$. The moduli space is given by $\mathbb{CP}^{\binom{n+d}{d}-1} = \mathbb{CP}^{19}$. The singular locus $\Sigma_{3,3}$ is a hypersurface in $\mathbb{CP}^{19}$ and the space of smooth cubic surfaces is given by $X_{3,3} = \mathbb{CP}^{19} \setminus \Sigma_{3,3}$. Theorem 4.1 tells us that all cubic surfaces are diffeomorphic to each other.

4.2 Cubic Surfaces and 27 lines

In Section 4.1 we used the universal bundle $U_{d,n} \xrightarrow{\pi} X_{d,n}; (X, p) \mapsto X$ to determine the diffeomorphism type of smooth degree d hypersurfaces in $\mathbb{CP}^n$.

We will discuss additional structures of fibers captured by this universal bundle for cubic surfaces in $\mathbb{CP}^3$. Every smooth cubic surface has exactly 27 lines on them, and this statement can be proved using a similar framework as Theorem 4.1: construct the incidence correspondence of lines on cubic surfaces, take a look at a particular fiber, and generalize that to every single fiber by the connectedness of the base space.

First, observe that lines on the Fermat cubic $X = V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subseteq \mathbb{CP}^3$ can be computed explicitly. Let $\omega = e^{2\pi i/3}$ be a third root of unity. There are 9 lines given by $(x_0 : \omega^i x_0 : x_2 : \omega^j x_2)$ for some $0 \leq i, j \leq 2$. Then by linear change of coordinates, there are 27 lines in total.

Lemma 4.1 (Fermat Cubic)**.** *The Fermat cubic $X = V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subseteq \mathbb{CP}^3$ contains exactly 27 lines.*

Now we construct the incidence correspondence of lines on cubic surfaces. Consider

$$M := \{(X, L) : L \text{ is a line on } X\} \subseteq X_{3,3} \times G(2,4),$$

where $G(2,4)$ is the Grassmannian of 2-dimensional planes in a 4-dimensional vector space, or equivalently lines in $\mathbb{CP}^3$. There is a natural projection $\pi : M \to X_{3,3}; (X, L) \mapsto X$, where the lines on a cubic surface is just its preimage under π.

Lemma 4.2. *The projection $\pi : M \to X_{3,3}; (X, L) \mapsto X$ is a covering.*

Proof. We prove the statement in two steps: First, we show that the incidence correspondence is closed in the Zariski topology of $X_{3,3} \times G(2,4)$, cut out by polynomials. Second, we show that the map $\pi : M \to X_{3,3}$ is a local diffeomorphism at line in $G(2,4)$.

For the first part, consider the affine chart U of $G(2,4)$, given by Plucker coordinates

$$\left\{ \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{pmatrix} : x, y, z, w \in \mathbb{C} \right\},$$

where we take the row spans of the matrices to represent planes in $\mathbb{C}^4$, i.e., lines in $\mathbb{CP}^3$.

Recall that we have coordinates (a_I) for $X_{3,3} \subseteq \mathbb{CP}^{19}$, given by the cubic polynomials $f = \sum_I a_I x^I$. So we have coordinates for the product $X_{3,3} \times G(2,4)$, denoted by (a_I, x, y, z, w).

The condition for a line to lie on a cubic surface $X = V(f_{a_I})$ is that the polynomial $f_{a_I}(s(1,0,x,y) + t(0,1,z,w)) = 0$ for all $s, t \in \mathbb{C}$. Expanding this expression, the coefficients of $s^3, s^2 t, s^1 t^2, t^3$ must all vanish, which gives us four polynomial conditions f_i in variables a_I, x, y, z, w.

Applying the implicit function theorem to polynomials f_i, we can check that $\pi : M \to X_{3,3}$ is a local diffeomorphism at $(x, y, z, w) = (0,0,0,0)$. By linear change of coordinates, we get that the map $\pi : M \to X_{3,3}$ is a local diffeomorphism at any $(x, y, z, w) \in \mathbb{C}^4$.

Now, for any compact subset $K \subseteq X_{3,3}$, we have that $\pi^{-1}(K)$ is Zariski closed in $K \times G(2,4)$, which is compact, so the projection π is proper. Combining these results, we know that $\pi : M \to X_{3,3}$ is a proper local diffeomorphism, and thus a covering map. □

Now we can leverage the topology of $\pi : M \to X_{3,3}$ as covering spaces to extract information about each of the fiber.

Theorem 4.2. *Every smooth cubic surface contains exactly 27 lines.*

Proof. The cardinality of elements in each fiber of $\pi : M \to X_{3,3}$ is locally constant. But $X_{3,3}$ is connected, so it is globally constant. Since there are exactly 27 lines on the Fermat cubic, we know that there are 27 lines on any smooth cubic surface. □

The incidence correspondence we constructed above only encodes the information of the number of lines. If we care about particular geometries, we can define incidence correspondences differently and use a similar trick as before to extend a property of a particular fiber to all fibers.

Lemma 4.3 (Fermat Cubic)**.** *Consider the Fermat cubic* $X = V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subseteq \mathbb{CP}^3$*. For any pair of skew lines* L_1, L_2 *in* X*, there are exactly* 5 *other lines in* X *meeting both* L_1 *and* L_2*.*

Then similarly, we can construct an appropriate incidence correspondence and prove the following.

Lemma 4.4. *For any smooth cubic surface* X*, any two pair of skew lines* L_1, L_2 *in* X *have exactly* 5 *other lines in* X *meeting both of them.*

Using this property, we will give a different desciption of smooth cubic surfaces, with which we can describe the lines and their intersection patterns explicitly.

Theorem 4.3. *A smooth cubic surface in* $\mathbb{CP}^3$ *is birational to* $\mathbb{CP}^2$*. In fact, it is isomorphic to* $\mathbb{P}^2$ *blow up at* 6 *points.*

Proof. Let X be a smooth cubic surface and let L_1, L_2 be a pair of skew lines on X.

Let $f : X \to L_1 \times L_2$ defined in the following way. If $a \in X \smallsetminus L_1$, let H be the unique plane in $\mathbb{CP}^3$ that contains L_1 and a and set $f_2(a) = H \cap L_2$. Similarly, define $f_1(a)$. Then we define $f(a) = (f_1(a), f_2(a))$. Note that $f_1(a)$ is a point on L_1 and $f_2(a)$ is a point on L_2 and a lies on the unique line that passes through these two lines. This gives a well-defined morphism $X \to L_1 \times L_2$. For the inverse, we define $L_1 \times L_2 \dashrightarrow X$ as mapping a pair of points $(a_1, a_2) \in L_1 \times L_2$ to the third intersection point of X with the line through a_1 and a_2. This is not well-defined exactly when the line through a_1 and a_2 is contained in X, and by Lemma 4.4, we get that there are exactly five such lines. At a special point (a_1, a_2), f will map the entire line passing through a_1 and a_2 to (a_1, a_2). Then we can take blow ups of $L_1 \times L_2$ at these five points and obtain an isomorphism $X \cong Bl_5(L_1 \times L_2)$.

Since $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at one point is isomorphic to $\mathbb{P}^2$ blown up in two points, $X \cong Bl_6\mathbb{P}^2$. □

Under this isomorphism $X \cong Bl_6\mathbb{P}^2$, we can give an explicit description of the second cohomology group of $H^2(X, \mathbb{Z}) \cong H^2(Bl_6\mathbb{P}^2, \mathbb{Z})$.

Proposition 4.1 ([Har77])**.** *The second cohomology group of* $H^2(Bl_6\mathbb{P}^2, \mathbb{Z})$ *is* 7*-dimensional, generated by* $l, e_1, \dots, e_6$ *where* l *represents the hyperplane class and* $e_1, \dots, e_6$ *represents the exceptional divisors from blowing up* $\mathbb{P}^2$*. Furthermore, these basis elements are mutually orthogonal with self-intersections* $l \cdot l = 1$ *and* $e_i \cdot e_i = -1$*.*

We also have a description of exceptional classes, $\mathcal{E}(S) = \{e \in H^2(Bl_6\mathbb{P}^2, \mathbb{Z}) : e \cdot e = -1, e \cdot -K_X = 1\}$. Expressed as a linear combination of basis elements $l, e_1, \ldots, e_6$, we have a complete list of $\mathcal{E}(S)$:

(a) e_i, $i = 1, \ldots, 6$.

(b) $l - e_i - e_j$, $1 \le i < j \le 6$.

(c) $2l - e_{i_1} - \ldots - e_{i_5}$ where $1 \le i_1 < \ldots < i_5 \le 6$.

Note that there are exactly 27 of these cohomology classes and this is not a coincidence. These are exactly represented by the 27 lines on the cubic surface.

Theorem 4.4. *[Har77, Theorem 4.9] The 27 lines on a cubic surface $X \cong Bl_6\mathbb{P}^2$ each has self-intersection -1 and they are the only irreducible curves with negative self-intersection on X. They are*

(a) the 6 exceptional lines given by $E_1, \ldots, E_6$.

(b) the strict transform of the $\binom{6}{2} = 15$ lines through two of the blown up points.

(c) the strict transform of the $\binom{6}{5} = 6$ conics through five of the blown up points.

4.3 The Monodromy Group of the 27 lines

Recall that we have the fiber bundle given by the universal family $U_{n,d} \to X_{3,3}$. We may ask about how smooth cubic surfaces deform in families. One way to describe this is through the monodromy representations: as we vary the cubic surfaces over a loop in $X_{3,3}$, we have an induced diffeomorphism and thus induced automorphisms on the cohomology groups. In particular, let's fix an arbitrary point $p \in X_{3,3}$ that represents $X \subseteq \mathbb{CP}^3$. Then for each $n \ge 2$ we have monodromy representations

$$\rho_n : \pi_1(X_{3,3}, p) \to \mathrm{GL}(H^n(X; \mathbb{Z})).$$

Since X is a smooth hypersurface in $\mathbb{CP}^3$, the monodromy representations ρ_n are trivial unless $n = 2$. So we will focus on the monodromy representation

$$\rho_2 : \pi_1(X_{3,3}, p) \to \mathrm{GL}(H^2(X; \mathbb{Z})).$$

Recall from Proposition 4.1 that the second cohomology of a cubic surface is given by $H^2(X; \mathbb{Z}) \cong \mathbb{Z}^7$, equipped with an intersection pairing that is a symmetric bilinear form of signature $(1, 6)$. The monodromy representations have to preserve this form, and thus $G := \rho_2(\pi_1(X_{3,3}, p))$ is a subgroup of $O(H^2(X; \mathbb{Z}))$.

Lemma 4.5. *The monodromy group G is a subgroup of $W(E_6)$.*

Proof. Since each surface in $X_{3,3}$ comes with a canonical embedding in $\mathbb{CP}^3$, any element in the monodromy must fix the the first Chern class of the anticanonical bundle, $-K_X$. Thus, $G \leq O((K_X)^\perp) \leq O(H^2(X;\mathbb{Z}))$. The orthogonal complement of K_X can be identified with the lattice $E_6 = (\mathbb{Z}K_X)^\perp \cong \mathbb{Z}^6$, where the identification is given by the root system $\{\alpha_0 = l - e_1 - e_2 - e_3, \alpha_1 = e_1 - e_2, \ldots, \alpha_5 = e_5 - e_6\}$. The exceptional classes in $\mathcal{E}(S)$, which are exactly represented by the 27 lines, correspond to certain roots in the E_6 lattice. We can further bound the group G by the induced action on $\mathcal{E}(S)$. Define the *Weyl group* $W(E_6)$ as the subgroup generated by reflections $s_{\alpha_i} : v \mapsto v + (v, \alpha_i)\alpha_i$. This group is isomorphic to the automorphism group of 27 lines respecting the intersection pairing. In particular, the generator s_{α_0} gives the quadratic transformation based at points $p_1, p_2, p_3 \in \mathbb{P}^2$ and the generators s_{α_i} interchange E_i with E_{i+1} for $1 \leq i \leq 5$. Thus $G \leq W(E_6)$. □

Theorem 4.5. *[Har79] The monodromy group $G \cong W(E_6)$.*

Proof. By Lemma 4.5, $G \leq W(E_6)$. Now we want to show that any element in $W(E_6)$ can be realized by monodromy. First, on a cubic surface X, $H^2(X, \mathbb{Z})$ is generated by any set of six skew lines. So the action of $W(E_6)$ on $H^2(X, \mathbb{Z})$ is fully determined by where we send the six lines. Note that there are 72 such unordered sets of six skew lines, and a cardinality count using orbit stabilier theorem tells us that $W(E_6)$ is generated by elements that fix a set of six skew lines. So it suffices to find loops in $X_{3,3}$ that inducing any permutation σ on a set of six skew lines.

Since X is identified with $\mathbb{P}^2$ blown up at 6 points $(p_1, \ldots, p_6)$, we can construct an explicit path between $(p_1, \ldots, p_6)$ and $(\sigma(p_1), \ldots, \sigma(p_6))$ in $\mathrm{Conf}_6(\mathbb{P}^2)$. For each t, we have $Bl_{\{p_i(t)\}_{i=1}^6}\mathbb{P}^2$. We can choose sections $\phi_0(t), \ldots, \phi_3(t)$ of the anticanonical bundles of $Bl_{\{p_i(t)\}_{i=1}^6}\mathbb{P}^2$ that vary continuously with respect to t, giving embeddings to $\mathbb{CP}^3$.

In this way, we constructed a loop in $X_{3,3}$ that induces the permutation σ on a set of six skew lines. □

Linking back to the degree 27 covering $M \to X_{3,3}$, we can see that the monodromy representation here is given by $\rho : \pi_1(X_{3,3}, p) \to S_27$, by permuting the 27 lines. The image of this representation exactly agrees with the monodromy group G of the universal family.

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