Quillen's Solution to Serre's Problem

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Week 4 Notes for AVB seminar

These notes are for a talk in eCHT seminar on algebraic vector bundles. We will start with the statements of Serre's problem and Horrocks' theorem which were introduced last week. Then we will move on to prove Quillen's Patching theorem and use it to extend the Horrocks' theorem from local rings to arbitrary rings. Lastly, we will give Quillen's proof of Serre's problem based on induction.

1 Serre's Problem and Local Horrocks' Theorem

Theorem 1.1 (Geometric local Horrocks)**.** *Let R be a local ring. If a vector bundle on $\mathbb{A}^1_R$ extends to $\mathbb{P}^1_R$ then it is trivial.*

Remark 1.1. *Consider the extension of scalars $\otimes_R R[t] : \mathrm{Mod}_R \to \mathrm{Mod}_{R[t]}$. When R is local, any finitely generated projective modules are free. This functor preserves direct sum, so it sends free R-modules to free $R[t]$-modules of the same rank. In addition, for any finitely generated $R[t]$-module, if it is indeed extended from an R-module, then it must be extended from a free R-module. Thus, a vector bundle on $\mathbb{A}^1_R$ if trivial iff the corresponding $R[t]$-module is extended from an R-module.*

This allows us to rephrase the Horrocks' theorem.

Theorem 1.2 (Geometric local Horrocks rephrased)**.** *Let R be a local ring. If a vector bundle on $\mathbb{A}^1_R$ extends to $\mathbb{P}^1_R$ then it is extended from* $\operatorname{Spec} R$.

This theorem gives us a way to tell if a vector bundle over $\mathbb{A}^1_R$ is trivial or not - we can just check if this vector bundle corresponds to an $R[t]$-module extended from R. Here we can see that we are phrasing this condition algebraically, which indeed makes things easier to work with. For the remaining of this section, we want to give an algebraic version of the local Horrocks' theorem.

Recall that $\mathbb{P}^1_R$ could be constructed by glueing two copies of $\mathbb{A}^1_R$. Let's denote these two affine lines by $\mathbb{A}^1_{R,0} := \mathrm{Spec}(R[t])$ and $\mathbb{A}^1_{R,\infty} := \mathrm{Spec}(R[s])$. Consider the ring homomorphisms

$$R[s] \to R[t,t^{-1}]; s \mapsto t^{-1};$$

$$R[t] \to R[t,t^{-1}]; t \mapsto t.$$

We may glue $\mathbb{A}^1_{R,0} = \mathrm{Spec}(R[t])$ and $\mathbb{A}^1_{R,\infty} = \mathrm{Spec}(R[s])$ along $\mathrm{Spec}(R[t,t^{-1}])$ and construct $\mathbb{P}^1_R$ as the pushout.

$$\begin{array}{ccc} \mathrm{Spec}(R[t,t^{-1}]) & \longrightarrow & \mathrm{Spec}(R[s]) = \mathbb{A}^1_{R,\infty} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R[t]) = \mathbb{A}^1_{R,0} & \longrightarrow & \mathbb{P}^1_R \end{array}$$

Since $\mathbb{A}^1_{R,\infty} \to \mathbb{P}^1_R$ and $\mathbb{A}^1_{R,\infty} \to \mathbb{P}^1_R$ give a Zariski cover, vector bundles over $\mathbb{P}^1_R$ could be specified by how the vector bundles on these two affine lines glue together.

This cover is great, but for the purpose of this talk we will consider another cover where we can get a more precise condition of when a vector bundle over $\mathbb{A}^1_R$ extends to $\mathbb{P}^1_R$.

Consider the covering of $\mathbb{P}^1_R$ given by $\{\mathbb{A}^1_{R,0} \to \mathbb{P}^1_R, \mathcal{O}_{\mathbb{P}^1_{R,\infty}} \to \mathbb{P}^1_R\}$ where $\mathcal{O}_{\mathbb{P}^1_{R,\infty}}$ is the stalk of the structural sheaf of $\mathbb{P}^1_R$ at infinity. In particular, $\mathcal{O}_{\mathbb{P}^1_{R,\infty}}$ is exactly the local ring $R[s]_{(m,s)}$ where m is the unique maximal ideal of the local ring R.

The intersection $\mathrm{Spec}(\mathcal{O}_{\mathbb{P}^1_{R,\infty}}) \cap \mathbb{A}^1_{R,0}$ could be derived algebraically. Consdier a distinguished open set $D(f) \subset \mathrm{Spec}(R[s])$ that contains the infinity point. Equilvalently, f does not live in the maximal ideal $(m,s) \subset R[s]$. In particular, the polynomial $f \notin (s)$, which implies that it has a non-vanishing constant term. Also, $f \notin m$, so the constant term must be a unit since all elements out of the maximal ideal of a local ring are units. Thus, when we intersect all the distinguished open sets containing the infinity point, on the level of functions we are inverting all the polynomials whose constant terms are units, which is the same as inverting all the polynomials with constant term 1. So, $\mathcal{O}_{\mathbb{P}^1_{R,\infty}} = (1 + sR[s])^{-1}R[s]$.

Note that the intersection $\operatorname{Spec}(\mathcal{O}_{\mathbb{P}^1_{R,\infty}}) \cap \mathbb{A}^1_{R,0}$ happens on $\operatorname{Spec}(R[t, t^{-1}])$, so we can look at the pullback of the following diagram to figure out what the intersection should be.

$$\begin{array}{ccc} & & \operatorname{Spec}(\mathcal{O}_{\mathbb{P}^1_{R,\infty}}) = \operatorname{Spec}\left((1 + sR[s])^{-1}R[s]\right) \\ & & \downarrow \\ \operatorname{Spec}(R[t, t^{-1}]) & \longrightarrow & \mathbb{A}^1_{R,\infty} = \operatorname{Spec} R[s] \end{array}$$

We may see that this pullback is exactly given by $\operatorname{Spec}(R[t, t^{-1}] \otimes_{R[s]} (1 + sR[s])^{-1}R[s])$. Tensoring with $\mathbb{R}[t, t^{-1}]$ over $R[s]$ is just formally setting the variable s in $(1 + sR[s])^{-1}R[s]$ to be t^{-1}, and thus inverting polynomials with constant term 1 in $R[s]$ is the same thing as inverting monic polynomials in $R[t]$. Therefore, $R[t, t^{-1}] \otimes_{R[s]} (1 + sR[s])^{-1}R[s]$ is the localization of $R[t]$ at all monic polynomials.

Let's denote the localization of $R[t]$ at all monic polynomials by $R\langle t\rangle$. Now we have the following diagram that illustrates the covering of $\mathbb{P}^1_R$.

$$\begin{array}{ccc} \operatorname{Spec}(R\langle t\rangle) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\mathbb{P}^1_{R,\infty}}) \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R[t]) = \mathbb{A}^1_{R,0} & \longrightarrow & \mathbb{P}^1_R \end{array}$$

It turns out that this is a faithfully flat cover and vector bundles form a fpqc stack. While details regarding these terns are swept under the rug, the upshot is that we can glue vector bundles over $\mathbb{A}^1_{R,0}$ and vector bundles over $\operatorname{Spec}(\mathcal{O}_{\mathbb{P}^1_R})$ to form vector bundles over $\mathbb{P}^1_R$ as long as they agree on the overlap.

With this covering, we may see that all vector bundles over $\operatorname{Spec}(\mathcal{O}_{\mathbb{P}^1_R})$ correspond to finitely generated projective modules over the local ring $\mathcal{O}_{\mathbb{P}^1_R}$, and thus must all be trivial.

If we want a vector bundle over $\mathbb{A}^1_{R,0}$ to agree with a vector bundle over $\operatorname{Spec}(\mathcal{O}_{\mathbb{P}^1_R})$, it must be trivial when restricted to $\operatorname{Spec}(R\langle t\rangle)$. Equivalently, an $R[t]$-module M must be free when we extend the scalars to $R\langle t\rangle$. In this way we get the algebraic version of the local Horrocks' theorem:

Theorem 1.3 (Algebraic Local Horrocks). *Let R be a local ring. Let $R\langle t\rangle$ be the localization of $R[t]$ at all monic polynomials. Let M be a finitely generated projective $R[t]$-module. Then M is free iff $M \otimes_{R[t]} R\langle t\rangle$ is free as a module over $R\langle t\rangle$.*

2 Quillen's Patching Theorem

In this section, we will introduce Quillen's Patching theorem, which is a local-global characterization of extended modules. Let's first explain what extension means here.

Definition 2.1. *We say an $R[t]$-module M is extended from R if there exists an R-module N such that $N \otimes_R R[t] \cong M$.*

Quillen's patching theorem essentially says that if we want to check if an $R[t]$-module M is extended from R, it suffices to check this condition when we localize M at all maximal ideals.

Theorem 2.1 (Quillen's Patching Theorem). *Let R be any ring and let M be a finitely presented $R[t]$-module. If $M_m \in \operatorname{Mod}_{R_m[t]}$ is extended from R_m for every m, then M is extended from R.*

Remark 2.1. *Before we prove this, I want to make a comment. What is so awesome about this theorem is that it allows us the patch together local information. Combined with local Horrocks' theorem, Quillen's Patching Theorem is essentially saying that if R is a ring such that Horrock's theorem holds for R_m for all maximal ideal $m \trianglelefteq R$ then Horrocks' theorem holds for R.*

Proof. First, let $Q(M) = \{f \in R \,:\, M_f \in \mathrm{Mod}_{R_f[t]} \text{ is extended from } R_f\}$.

The proof comes in two steps. We first prove that $Q_R(M)$ is actually an ideal of R and then prove that this is in fact the unit ideal.

<u>Claim</u>: $Q_R(M)$ is an ideal of R.

By definition of an ideal, we need to show that $Q_R(M)$ is closed under absorption and addition.

Let $f \in Q_R(M)$ and let $r \in R$. We want to prove that $rf \in Q_R(M)$.

Note that we have the following commutative diagram since localization commutes with tensor products.

$$\begin{array}{ccc} \mathrm{Mod}_{R_f} & \xrightarrow{\otimes_{R_f} R_f[t]} & \mathrm{Mod}_{R_f[t]} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{R_{rf}} & \xrightarrow{\otimes_{R_{rf}} R_{rf}[t]} & \mathrm{Mod}_{R_{rf}[t]} \end{array}$$

Consider $M_{rf} \cong (M_f)_r \in \mathrm{Mod}_{R_{rf}[t]}$. Along localizations, we can always find an $R_f[t]$-module M_f that is mapped to it. Since $f \in Q_R M$, M_f is extended from R_f, i.e., there exists an R_f-module that is extended to M_f. This R_f-module must be given by M_f/tM_f. Then if we localize M_f/tM_f at the element $r \in R$, by the commutative diagram above we obtain the isomorphism

$$(M_f/tM_f)_r \otimes_{R_{rf}} R_{rf}[t] \cong M_{rf}.$$

Thus $Q_R(M)$ is closed under absorption.

Now we want to prove that it's also closed under addition.

Suppose $f, g \in Q_R(M)$. We want to show that $f + g \in Q_R(M)$ as well.

Let $S = R_{f+g}$ and $M' = M_{f+g} \in \mathrm{Mod}_{R_{f+g}} = \mathrm{Mod}_S$.

By absorption, $f(f+g) \in Q_R(M)$, which means $M_{f(f+g)} = M'_f$ is extended from $R_{f(f+g)} = S_f$. So, $f \in Q_S(M')$. Similarly, $g \in Q_S(M')$.

Note that M' is extended from S if and only if M_{f+g} is extended from R_{f+g}, which is true if and only if $f + g \in Q_R(M)$.

So it suffices to prove that M' is extended from S.

Since $S = R_{f+g}$, $D(f)$ and $D(g)$ form a Zariski cover for $\mathrm{Spec}(S[t])$ and we may glue modules together along the cover as long as they overlap on the intersection. In specific, we have an equilvalence of categories $\mathrm{Mod}_{S[t]} \cong \mathrm{Mod}_{S_f[t]} \times_{S_{fg}[t]} \mathrm{Mod}_{S_g[t]}$, where the category on the right is what we call the category of descent data for $S[t]$-modules along the Zariski cover given by $S_f[t]$ and $S_g[t]$. In other words, we have the following (homotopy) pullback diagram.

$$\begin{array}{ccc} \mathrm{Mod}_{S[t]} & \longrightarrow & \mathrm{Mod}_{S_f[t]} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{S_g[t]} & \longrightarrow & \mathrm{Mod}_{S_{fg}[t]} \end{array}$$

Since $f, g \in Q_S(M')$, M'_f is extended from S_f and M'_g is extended from S_g. So, there exists an S_f-module P_f such that $P_f \otimes_{S_f} S_f[t] \cong M'_f$ and an S_g-module P_g such that $P_g \otimes_{S_g} S_g[t] \cong M'_g$. While M'_g and M'_f both come from localizations of $M' \in \mathrm{Mod}_{S[t]}$, it is not guaranteed the isomorphisms can be glued together. However, if we modify these isomorphisms a little (details skipped here), we can actually glue the isomorphisms together along $S_{fg}[t]$ and get an isomorphism $P \otimes_S S[t] \cong M'$. Thus M' is extended from M and equilvalently $f + g \in Q_R(M)$.

Now we have proved that $Q_R(M)$ is an ideal in R. We want to show that it is in fact the unit ideal.

<u>Claim</u>: $Q_R(M) = R$.

Let N be the quotient M/tM.

By the assumption of Quillen's Patching Theorem, M_m is extended from R_m for all maximal ideal m in R.

Let m be an arbitrary maximal ideal. We have isomorphisms $\phi_m : M_m \cong (M_m/tM_m) \otimes_{R_m} R_m[t]$. Note that the module $(M_m/tM_m) \otimes_{R_m} R_m[t] \cong (M/tM)_m \otimes_{R_m} R_m[t] \cong (N \otimes_R R[t])_m$

since localization commutes with tensor product and quotient.

By Lam I.2.16, we know that ϕ_m is the further localization of some $\phi_g : M_g \cong (N \otimes_R R[t])_g$ for some g. In other words, ϕ_m comes from the restriction of an isomorphism on a disdinguished open set $D(g)$.

From this isomorphism ϕ_g, we see that $g \in Q_R(M)$. Since $g \notin m$, $Q_R(M) \notin m$.

This argument above holds for all maximal ideals, so $Q_R(M)$ is not contained by any maximal ideal of R and thus must be the unit ideal. This implies that $1 \in Q_R(M)$.

Therefore, $M_1 \cong M$ is extended from $R_1 = R$. □

3 Affine Horrocks

Next we will use Quillen's Patching theorem to generalize local Horrocks' theorem.

Theorem 3.1 (geometric Affine Horrocks). *Let P be an vector bundle over $\mathbb{A}^1_1$. If P extends to $\mathbb{P}^1_R$, then it is extended from R.*

Proof. Since P extends from $\mathbb{A}^1_R \to \mathbb{P}^1_R$, P_m extends from $\mathbb{A}^1_{R_m} \to \mathbb{P}^1_{R_m}$. By local Horrocks, $P_m \in \mathrm{Mod}_{R_m[t]}$ is extended from R_m. By Quillen's Patching Theorem, $P \in \mathrm{Mod}_{R[t]}$ is extended from R. □

Equivalently we have an algebraic version of this.

Theorem 3.2 (algebraic Affine Horrocks). *Let R be any ring. Let P be a a finitely generated projective modules over $R[t]$. If $R \otimes_{R[t]} R\langle t\rangle$ is free over $R\langle t\rangle$, then P is extended from R.*

4 Quillen's Solution to Serre's Problem

In this section, we will prove Serre's Porblem.

We will first state an essential fact that will show up in the proof.

Lemma 4.1. *When R is a PID, $R\langle t\rangle$ is also a PID.*

Now we are ready to prove Serre's Problem!

Theorem 4.1. *Let R be a PID, then every finitely generated projective $R[t_1, \ldots, t_n]$-module is free.*

Quillen's proof. The proof induces on n, the number of variables in the polynomial ring.

Base case: The statement is true for $n = 0$, since every finitely generated projective module over PID is free.

Inductive hypothesis: Suppose all finitely generated projective modules over $A = R[t_1, \ldots, t_{n-1}]$ are free whenever R is a PID.

Inductive step: Let P be a finitely generated $A[t_n]$-module. Then we have the following localizations of rings and corresponding localizatiaons of modules:

$$A[t_n] \subseteq R\langle t_n\rangle[t_1, \ldots, t_{n-1}] \subseteq A\langle t_n\rangle$$

$$P \mapsto P \otimes R\langle t_n\rangle[t_1, \ldots, t_{n-1}] \mapsto P\langle t_n\rangle$$

Note that $P \otimes R\langle t_n\rangle[t_1, \ldots, t_{n-1}]$ is free by inductive hypothesis since $R\langle t_n\rangle$ is a PID. Also, $P \otimes R\langle t_n\rangle[t_1, \ldots, t_{n-1}] \mapsto P\langle t_n\rangle$ is a further localization where monic polynomials in t_n with coefficeints in A are also inverted. So $P\langle t_n\rangle$ is free over $A\langle t_n\rangle$. By Horrocks' theorem, P is extended from A. Thus P is a free module over R.

This finishes the induction and proves the Serre's problem. Hooray! □