MILNOR NUMBERS OF COMPLEX HYPERSURFACES

Zhong Zhang

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Supervisor of Thesis
Mona Merling
Assistant Professor of Mathematics

Graduate Group Chair
Ron Donagi
Thomas A. Scott Professor of Mathematics

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Abstract

Milnor's study of singular complex hypersurfaces, including the renowned fibration theorem, give us a rich understanding of isolated hypersurface singularities from a topological perspective. In this thesis, we will take a closer look at Milnor's findings and explore fascinating ideas that have emerged from them.

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1 Introduction

In 1956, Milnor [Mil56] discovered 7-dimensional exotic spheres, smooth 7-manifolds homeomorphic but not diffeomorphic to the standard 7-spheres. Interestingly, these exotic spheres are closely related to singular complex hypersurfaces, showing up as links of isolated singular points. Brieskorn [Bri66a] proved that every exotic sphere of dimension > 6 that bounds a parallelizable manifold is diffeomorphic to the link of a singularity.

This motivated Milnor's study of singular complex hypersurfaces, including the celebrated fibration theorem. His studies give us a rich understanding of isolated hypersurface singularities. We start in Section 2 with Milnor's classical findings and give interpretations of the Milnor number of an isolated singularity from topological and algebraic perspectives. In Section 3, we show Kouchnirenko's estimates of the Milnor number using Newton Polyhedra [Kou76]. In Section 4, we illustrate how Milnor numbers behave under deformations of hypersurfaces. We continue with a discussion of a particular family of hypersurfaces $\{f^{-1}(t) : t \in \mathbb{C}\}$ in Section 5 and conclude with a generalization of Milnor numbers to non-isolated singularities given by Parusiński in Section 6.

2 Classical Milnor numbers

2.1 The Milnor fibration

Consider a polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$. The *vanishing locus of* f is defined as $V(f) := \{p \in \mathbb{C}^{n+1} \mid f(p) = 0\}$. Such a set is called a *complex hypersurface*. A hypersurface may have *singular* points, which are the points in $V(f)$ where the gradient $\nabla f = \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)$ vanishes. These points are of particular interest to us, because their local topology is distinct from non-singular points on the hypersurface.

If $p \in V(f)$ is not singular, then the partial derivatives of f do not all vanish at the point p. Then $\nabla f(p) = \left(\frac{\partial f}{\partial x_0}(p), \dots, \frac{\partial f}{\partial x_n}(p)\right)$, viewed as a $1 \times (n+1)$ matrix, has full rank. So, we can apply the implicit function theorem to conclude that $V(f)$ locally looks like the graph of a continuous function and hence must be a real $2n$-dimensional manifold. But at singular points, all partial derivatives vanish and we cannot draw any conclusion using the implicit function theorem.

Brauner [Bra28] used the following construction to study the special case when f defines a plane curve, i.e., $f \in \mathbb{C}[x_0, x_1]$: Suppose the plane curve $V(f)$ has a singular point p. The local topology of $V(f)$ at p could be studied by taking small 3-spheres centered at p and looking at their intersections with $V(f)$. In this way, we are taking "slices" of $V(f)$ through the lens of 3-spheres of various sizes.

Milnor generalized this idea and introduced the Milnor fibration for singular algebraic hypersurfaces of arbitrary dimension. For the remainder of this section, most results come from Milnor's book [Mil68].

For a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$, one may consider a small sphere S_ε of radius ε centered around a singular point. That is, let $S_\varepsilon = \{q \in \mathbb{C}^{n+1} : \|q - p\| = \varepsilon\}$ be a $(2n+1)$-dimensional sphere of radius ε. Milnor proved that for $\varepsilon > 0$ sufficiently small, S_ε intersects $V(f)$ transversally. Since $V(f)$ is a $2n$-dimensional algebraic set over $\mathbb{R}$ and S_ε is of codimension 1, the intersection $K_\varepsilon := S_\varepsilon \cap V(f)$ is an algebraic set of codimension 3. The information about the topology of the hypersurface at the singular point is encoded in the way that K_ε embeds in S_ε.

Example 2.1. Let $f = z^m + w^n \in \mathbb{C}[z, w]$ for some coprime natural numbers m and n. Then $S_\varepsilon = \{(z, w) \in \mathbb{C}^2 : \|z\|^2 + \|w\|^2 = \varepsilon^2\}$ and $V(f) = \{(z, w) : z^m + w^n = 0\}$. Let $z = re^{i\phi_1}$ and $w = se^{i\phi_2}$ where $r, s \in \mathbb{R}$ and $\phi_1, \phi_2 \in [0, 2\pi)$. If $(z, w) \in K_\varepsilon = S_\varepsilon \cap V(f)$, then $r^2 + s^2 = \varepsilon^2$ and $z^m + w^n = 0$. This implies that $r^m e^{im\phi_1} = -s^n e^{in\phi_1}$, and thus $e^{im\phi_1} = -e^{in\phi_1}$. Hence, the space K_ε would be a torus knot of type (m, n), lying in the torus parametrized by $(re^{i\phi_1}, se^{i\phi_2})$.

In the case where $n = 1$, the space K_ε may not be a connected object. For example, the polynomial $f = x_1^3 + x_2^6$ illustrates that K_ε consists of two unknotted circles. However, when $n \geq 2$, the space K_ε is guaranteed to be connected. Using Morse theory, Milnor demonstrated that K_ε is $(n-2)$-connected.

Here is an example of higher dimension given by Brieskorn: [Bri66b]

Example 2.2. Consider $f_r = x_0^3 + x_1^{6r-1} + x_2^2 + x_3^2 + x_4^2 \in \mathbb{C}[x_0, \ldots, x_4]$, where $r \in \{1, 2, \ldots, 28\}$. Then, $S_\varepsilon \cap V(f_r)$ gives 28 spheres that are homeomorphic to the standard unit 7-sphere S^7, but with different smooth structures.

With the aforementioned construction and for any singular point on $V(f)$, Milnor defined what we now call the *Milnor fibration*:

$$\begin{aligned} M_f \colon S_\varepsilon \setminus K_\varepsilon &\to S^1 \\ z &\mapsto \frac{f(z)}{|f(z)|}. \end{aligned}$$

Milnor used the Curve Selection Lemma [Mil68] to show that the map M_f above has no critical points, which demonstrates that the fibers of this map are smooth manifolds. From there he constructed a tangential vector field on $S_\varepsilon \setminus K_\varepsilon$ and established that its behavior is under control when approaching the boundary of the fibers, K_ε:

Lemma 2.3. *There exists a smooth tangential vector field w on $S_\varepsilon \setminus K_\varepsilon$ which satisfies the following conditions:*

- *the real part of the inner product $\langle w(z), i \operatorname{grad} \log f(z) \rangle$ is identically 1,*
- *the imaginary part satisfies $|Re \langle w(z), \operatorname{grad} \log f(z) \rangle| < 1$.*

Then we may consider the trajectories of the differential equation $\frac{dz}{dt} = w(z)$. The two conditions in the lemma above guarantee that solution curves of this differential equation cannot tend towards K_ε as t tends towards some finite limit t_0.

Lemma 2.4. *Given $z_0 \in S_\varepsilon \setminus K_\varepsilon$, there exists a unique smooth path $p \colon \mathbb{R} \to S_\varepsilon \setminus K_\varepsilon$ which satisfies the differential equation $\frac{dp(t)}{dt} = w(p(t))$ with the initial condition $p(0) = z_0$.*

For each t, the path $p(t)$ defines a diffeomorphism on $S_\varepsilon \setminus K_\varepsilon$ which carries fibers of the Milnor fibration to other fibers. Define

$$F_\theta := M_f^{-1}(e^{i\theta}) = \{z \in S_\varepsilon \setminus K_\varepsilon : f(z) = e^{i\theta}\}.$$

Then the path $p(t)$ takes the fiber F_θ to the fiber $F_{\theta+t}$. This gives the fibration theorem.

Theorem 2.5 (Milnor's fibration theorem)**.** *The space $S_\varepsilon \setminus K_\varepsilon$ is a smooth fiber bundle over S^1, given by the Milnor fibration $M_f \colon S_\varepsilon \setminus K_\varepsilon \to S^1$.*

Milnor showed that the fiber of this map, F_θ, is a smooth $2n$-dimensional parallelizable manifold with the homotopy type of a finite CW-complex of dimension n. Furthermore, it is $(n-2)$-connected.

F_θ is related to the local topology of the singularity by the following theorem.

Theorem 2.6. *If the complex number $c \neq 0$ is sufficiently close to zero, then the intersection of the complex hypersurface $f^{-1}(c)$ and the open ε-ball B_ε centered at p, denoted as $f^{-1}(c) \cap B_\varepsilon$, forms a smooth manifold diffeomorphic to F_θ.*

So, the fiber of the Milnor fibration gives us information about level sets of f that are close to $V(f) = f^{-1}(0)$. In other words, if we perturb f by a sufficiently small constant $c \in \mathbb{C}^*$, then the corresponding hypersurface locally is diffeomorphic to the fiber F_θ of the Milnor fibration.

This leads to an equivalent version of Milnor's fibration theorem, which shows up frequently in the literature.

Theorem 2.7 (fibration theorem, second version)**.** *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ and suppose that $V(f)$ has a singular point p. Then, there exists $0 < \delta \ll \varepsilon$ such that for all $t \in D_\delta$, the restriction of f to a map $f^{-1}(\partial D_\delta) \cap B_\varepsilon \to \partial D_\delta$ is a smooth fiber bundle, where B_ε is the open ball of radius ε centered at the singularity point p and D_δ denotes the closed disc of radius δ centered at the origin in $\mathbb{C}$. Moreover, the diffeomorphism type of this fiber bundle is independent of ε and δ as long as they are sufficiently small.*

This fibration $f^{-1}(\partial D_\delta) \cap B_\varepsilon \to \partial D_\delta$ could be carried to the Milnor fibration $S_\varepsilon \setminus K_\varepsilon \to S^1$ by a smooth flow, which demonstrates the equivalence of these two fibrations.

We will mostly follow Milnor's notation in his book and allude to this equivalent version when necessary.

2.2 Milnor fiber at an isolated singularity

For the rest of this section, we will add the additional assumption that p is an isolated singularity.

We call p an *isolated singularity* if it has a neighborhood in $\mathbb{C}^{n+1}$ containing no other singular points. In that case, K_ε is a smooth submanifold for ε small enough.

In this case, Milnor proved the following result:

Lemma 2.8. *Let f be a complex polynomial with an isolated singularity $p \in V(f)$. The topology of the diffeomorphism type of the manifold K_ε is independent of the choice of ε as long as it is sufficiently small.*

Proof. If we consider the function $d\colon (V(f) \cap B_\varepsilon) \setminus \{p\} \to (0, \varepsilon)$ that measures the distance to the isolated singularity p, we can see that this is a proper smooth function. Furthermore, it is submersive and surjective. By Ehresmann's lemma, the function d is a locally trivial fibration, which shows that K_ε is independent of ε. □

In the case of isolated singularities, a stronger description of the fiber is given: Using Morse theory, Minor [Mil68] showed that the fiber F_θ must be $(n-1)$ connected. Combining this with some results from the last subsection, we know the homotopy type of F_θ:

Theorem 2.9. *F_θ is homotopy equivalent to a finite bouquet of spheres $\vee_i S^n$.*

Proof. In the previous subsection, it was established that F_θ is a parallelizable manifold with a dimension of $2n$ and homotopy type equivalent to a finite CW complex with dimension n. As a result, the cohomology group of dimension $n+1$ vanishes. Then by Poincaré Duality, the homology group of dimension $n-1$ is also trivial. Therefore, $H_n(F_\theta)$ must be finitely generated free abelian.

Since F_θ is $(n-1)$-connected, we can use the Hurewicz theorem to deduce that $\pi_n(F_\theta) \cong H_n(F_\theta)$ must be finitely generated free abelian for $n \geq 2$. This implies that we have a based map $S^n \vee \ldots \vee S^n \to F_\theta$, where each $S^n \to F_\theta$ corresponds to a generator of $\pi_n(F_\theta)$. Note that the map $S^n \vee \ldots \vee S^n \to F_\theta$ induces homology isomorphisms. By the generalized Whitehead theorem, we know that the map is in fact a homotopy equivalence.

For $n = 1$, F_θ is homotopy equivalent to a path-connected finite CW complex of dimension 1, which is a finite graph. Then consider a maximal tree in this graph, which exists because F_θ is path-connected. There exists a retract that exactly maps this tree to a 0-cell, that is, a vertex. The image of this retract is exactly a finite bouquet of loops. Hence, $F_\theta \simeq \vee_i S^1$. $\square$

Furthermore, Milnor noticed that Smale's h-cobordism theorem in high dimensions would imply that the closure of F_θ is actually diffeomorphic to the connected sum of a $2n$-ball with μ handles of middle dimension, if $n \neq 2$. This dimension gap was filled by Lê and Perron [TP79] later using another construction, which actually works in all dimensions.

The following definition counts the number of spheres in the bouquet $S^n \vee \ldots \vee S^n$.

Definition 2.10. Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ with an isolated singularity $p \in V(f)$ Then the *Milnor number* $\mu_p(f)$ of f at p is

$$\mu_p(f) := \operatorname{rank} H_n(F_\theta).$$

Remark 2.11. The Milnor number $\mu_p(f)$ is related to the Euler characteristic of F_θ. The Euler characteristic of F_θ is defined to be

$$\chi(F_\theta) = \sum (-1)^i \operatorname{rank} H_i(F_\theta).$$

Since F_θ only has nontrivial homology groups in degree 0 and n,

$$\chi(F_\theta) = 1 + (-1)^n \operatorname{rank} H_n(F_\theta) = 1 + (-1)^n \mu_p(f).$$

Example 2.12. Let $f = z^m + w^n \in \mathbb{C}[z,w]$. The fiber F_θ is a Seifert surface with boundary K_ε, which we computed to be a knot of type (m,n). The genus of this knot is given as $\frac{(m-1)(n-1)}{2}$ by knot theory, and thus

$$\chi(F_\theta) = 2 - 2 \cdot \frac{(m-1)(n-1)}{2} - 1 = 1 - (m-1)(n-1).$$

Then by the remark above, the Milnor number of f at the singular point 0 is

$$\mu_0(f) = (-1)^1(\chi(F_\theta) - 1) = (-1)(1 - (m-1)(n-1) - 1) = (m-1)(n-1).$$

2.3 Topological interpretation of the Milnor number

Next we take a slight detour through the notion of topological degree, which will give us an interesting interpretation of the Milnor number at isolated singular points.

Let $g\colon S^n \to S^n$ with $n > 0$. The induced map on the n-th homology groups $g_*\colon H_n(S^n) \to H_n(S^n)$ is a homomorphism of the form $g_*(x) = dx$ for some $d \in \mathbb{Z}$ depending only on g. In this way, one can associate an integer to each map from S^n to S^n. Since homotopy equivalent maps induce the same homology homomorphisms, this association gives rise to a well-defined degree map $\deg\colon [S^n, S^n] \to \mathbb{Z}$.

Extending from this degree map between spheres, one can define a notion of local degree of maps between manifolds of the same dimension.

Definition 2.13. Let M and N be compact oriented n-dimensional manifolds and let $f\colon M \to N$ be a smooth map. Consider a point $q \in N$ whose preimage $f^{-1}(q)$ is a finite collection of points. Let $p \in f^{-1}(q)$. Take an open ball V centered at q.

Suppose U is a sufficiently small open ball centered at p so that its image under f is completely contained by V and $U \cap f^{-1}(q) = \varnothing$. Then, f induces a map on the homology: $f_*\colon H_n(U, U \setminus \{p\}) \to H_n(V, V \setminus \{q\})$.

Since $H_n(U, U \setminus \{p\}) \cong H_n(V, V \setminus \{q\}) \cong H_n(S^n)$, using the excision and long exact sequence of homology groups, we have the following commutative diagram:

$$\begin{array}{ccc} H_n(U, U \setminus \{p\}) & \xrightarrow{f_*} & H_n(V, V \setminus \{q\}) \\ \downarrow{\cong} & & \downarrow{\cong} \\ H_n(S^n) & \longrightarrow & H_n(S^n). \end{array}$$

The *local degree* of f at p, which we call $\deg_p^{top}(f)$, is defined as the image of 1 under the induced map from $H_n(S^n)$ to $H_n(S^n)$.

Now let's return to the discussion of Milnor numbers.

Recall our assumption that p is an isolated singularity of $V(f)$, which means it is a zero of the function

$$\begin{aligned} \nabla f\colon \mathbb{C}^{n+1} &\to \mathbb{C}^{n+1}; \\ x &\mapsto \left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right). \end{aligned}$$

If we take an open ball in the codomain at 0 and a corresponding isolating open ball in the domain at p, we can follow the procedure mentioned above to compute the local degree of ∇f at p. Roughly speaking, the local topological degree captures how many times a local neighborhood of the domain wraps around a local neighborhood of the codomain.

The connection between the local degree and the Milnor number is given by the following theorem proven by Milnor.

Theorem 2.14. *The Milnor number of f at the point p equals the local degree of ∇f at p. That is, $\mu_p(f) = \deg_p^{top}(\nabla f)$.*

The basic idea behind the proof given by Milnor is that the the degree of ∇f is related to the Euler characteristic of F_θ by a formula, giving us a bridge to the the Milnor number.

This theorem provides us with an interesting interpretation of the Milnor number using the local topological degree: If we perturb ∇f slightly, then p should split up into a cluster of $\mu_p(f)$ points. In this sense, the Milnor number of p measures the degeneracy of the singularity of $V(f)$ at p. In particular, if the singularity is nondegenerate, which means the Jacobian matrix of ∇f is invertible at p, then $\mu_p(f) = 1$.

Example 2.15. Consider $f(z, w) = z^m + w^n \in \mathbb{C}[z, w]$. The gradient of the function is given by

$$\nabla f \colon \mathbb{C}^2 \to C^2;$$

$$(z, w) \mapsto (mz^{m-1}, nz^{n-1}).$$

The local degree of the map

$$\phi \colon \mathbb{C}^1 \to \mathbb{C}^1;$$

$$z \mapsto mz^{m-1}$$

is measured by the image of the generator 1 under the induced map on the second cohomology group of S^2:

$$H_2(S^2;\mathbb{Z}) \to H_2(S^2;\mathbb{Z});$$

$$1 \mapsto \deg_0^{top}(\phi) = m-1.$$

Similarly, for the map

$$\psi\colon \mathbb{C}^1 \to \mathbb{C}^1;$$

$$w \mapsto nw^{n-1},$$

we have

$$H_2(S^2;\mathbb{Z}) \to H_2(S^2;\mathbb{Z});$$

$$1 \mapsto \deg_0^{top}(\psi) = n-1.$$

Since $H_2(S^2;\mathbb{Z})$ is finitely generated and free, we can take coefficients in $\mathbb{R}$ instead. Then the Kunneth formula for the homology of relative CW pairs helps us compute the local degree of the map ∇f at 0: there exists a natural isomorphism

$$\tilde{H}_2(S^2;\mathbb{R}) \otimes_{\mathbb{R}} \tilde{H}_2(S^2;\mathbb{R}) \to \tilde{H}_4(S^2 \wedge S^2;\mathbb{R});$$

$$1 \otimes 1 \mapsto 1.$$

So we have the following commutative diagram:

$$\begin{tikzcd}
\tilde{H}_4(S^2 \wedge S^2;\mathbb{R}) \arrow[r, "(\nabla f)_*"] \arrow[d, "\cong"'] & \tilde{H}_4(S^2 \wedge S^2;\mathbb{R}) \arrow[d, "\cong"] \\
\tilde{H}_2(S^2;\mathbb{R}) \otimes_{\mathbb{R}} \tilde{H}_2(S^2;\mathbb{R}) \arrow[r, "\phi_* \otimes \psi_*"'] & \tilde{H}_2(S^2;\mathbb{R}) \otimes_{\mathbb{R}} \tilde{H}_2(S^2;\mathbb{R}).
\end{tikzcd}$$

The local degree of the map $\nabla f\colon \mathbb{C}^2 \to \mathbb{C}^2$ is determined by the image of 1 in the induced map $\tilde{H}_2(S^2 \wedge S^2; \mathbb{R}) \to \tilde{H}_2(S^2 \wedge S^2; \mathbb{R})$.

It follows from the diagram that

$$(\nabla f)_*(1) = \phi_*(1)\psi_*(1) = \deg_0^{top}(\phi)\deg_0^{top}(\psi) = (m-1)(n-1).$$

Therefore, we can conclude that $\deg_0^{top}(\nabla f) = (m-1)(n-1)$, which is consistent with our previous computation using the Euler characteristic.

2.4 Algebro-geometric interpretation of the Milnor number

With some basic tools in algebraic geometry, the Milnor number at a singularity has another interpretation.

If a point p is an isolated singularity of a complex hypersurface $V(f)$, then

$$p \in V(\nabla f) = V\left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right) = V\left(\frac{\partial f}{\partial x_0}\right) \cap \dots \cap V\left(\frac{\partial f}{\partial x_n}\right).$$

Since we assumed that p is an isolated singularity, we may see that the partial derivatives can't share a common component and thus the intersection $V\left(\frac{\partial f}{\partial x_0}\right) \cap \dots \cap V\left(\frac{\partial f}{\partial x_n}\right)$ in $\mathbb{C}^{n+1}$ is expected to be finite. Then, we can count the number of intersection points contributed by the point p: the intersection multiplicity at p is given by the dimension of the local algebra at p, $\dim_{\mathbb{C}}\left(\mathbb{C}[x_0, \dots, x_n] \middle/ \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)\right)_{\mathfrak{m}_p}$, where $\mathfrak{m}_p$ is the maximal ideal corresponding to the point p.

We can understand the dimension of the local algebra in the following way: First, since localization commutes with quotients,

$$\left(\mathbb{C}[x_0,\ldots,x_n]\Big/\left(\frac{\partial f}{\partial x_0},\ldots,\frac{\partial f}{\partial x_n}\right)\right)_{\mathfrak{m}_p} \cong \left(\mathbb{C}[x_0,\ldots,x_n]_{\mathfrak{m}_p}\Big/\left(\frac{\partial f}{\partial x_0},\ldots,\frac{\partial f}{\partial x_n}\right)\right).$$

We may see that the localization $\mathbb{C}[x_0,\ldots,x_n]_{\mathfrak{m}_p}$ contains a unique maximal ideal $\mathfrak{m}_p$, then the multiplicity of $V(\nabla f)$ at p is the largest integer μ such that $\nabla f \in \mathfrak{m}_p^{\mu}$ and $\nabla f \notin \mathfrak{m}_p^{\mu+1}$. Since f is described by a complex polynomial with respect to complex coordinates, we can write $f(x) = f_0 + f_1(x) + \ldots + f_k(x)$ with $f_i(x)$ homogeneous of degree i. The hypersurface $V(f)$ contains p if $f_0 = f(p) = 0$ and the intersection multiplicity can be described concretely: if $f_0 = \ldots = f_{\mu-1} = 0$ and $f_\mu \neq 0$, then μ is the multiplicity of $V(f)$ at p. In short, the multiplicity is defined as the lowest degree in the power series expansion of f at $p \in \mathbb{C}^n$.

Milnor [Mil68] and Palomodov [Pal67] showed that this algebraic definition of intersection multiplicity agrees with the topological degree, which gives rise to the following theorem.

Theorem 2.16. *The Milnor number of f at p equals the intersection multiplicity of the vanishing loci of the partial derivatives at the point p. That is,*

$$\mu_p(f) = \dim_{\mathbb{C}}\left(\mathbb{C}[x_0,\ldots,x_n]\Big/\left(\frac{\partial f}{\partial x_0},\ldots,\frac{\partial f}{\partial x_n}\right)\right)_{\mathfrak{m}_p}.$$

So the Milnor number reflects the number of algebraically independent directions at the singular point of the hypersurface.

Example 2.17. We refer back to the example in the last subsection: Let $f = z^m + w^n \in \mathbb{C}[z,w]$ for coprime natural numbers $m, n \geq 2$. Then the intersection multiplicity of

$V(\frac{\partial f}{\partial z})$ and $V(\frac{\partial f}{\partial w})$ at 0 is given by

$$\begin{aligned} &\dim_{\mathbb{C}} \left(\mathbb{C}[z,w] / \left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial w} \right) \right)_{(z-0,w-0)} \\ &= \dim_{\mathbb{C}} \left(\mathbb{C}[z,w] / \left(mz^{m-1}, nz^{n-1} \right) \right)_{(z,w)} \\ &= (m-1)\,(n-1)\,. \end{aligned}$$

The theorem above tells us that the Milnor number at 0 is $\mu_0(f) = (m-1)(n-1)$.

Remark 2.18. One benefit of this algebraic interpretation of the Milnor number is that it enables easy computations, as we can see from the example above.

3 Newton polyhedra and Milnor numbers

With the Milnor numbers defined in the previous section, we may ask whether we can estimate the Milnor number of an isolated singularity directly from the expression of the polynomial.

Kouchinrenko [Kou76] provided an answer to this question by utilizing a combinatorial object known as the Newton polyhedron. From there he defined the Newton number and illustrated that this number serves as a lower bound for the Milnor number. His results apply to more general functions like Laurent series or formal power series, where the Milnor number is allowed to be infinite. For our purposes, we will only focus on complex polynomials. Nevertheless, the definitions and results presented in this section can be extended to include series.

We start with the definition of the Newton polyhedron.

Definition 3.1. Consider a polynomial $f = \sum_{I \in \mathbb{N}^n} a_I x^I$. The *support* of f is $\operatorname{supp} f := \{I \in \mathbb{N}^n : a_I \neq 0\}$. We define the *Newton polyhedron* $\Gamma_+(f)$ of f as the convex hull of $\operatorname{supp} f$.

Then let the *Newton boundary* of f, denoted $\Gamma(f)$, to be the union of the compact faces of $\Gamma_+(f)$ that one can "view" from the origin, i.e., the union of all faces γ of $\Gamma_+(f)$ such that $\Gamma_+(f) \cap (\operatorname{conv}(\{0\} \cup \gamma)) = \gamma$. Let $\Gamma_-(f)$ denote the polyhedron $\bigcup_{\gamma \in \Gamma(f)} \operatorname{conv}(\{0\} \cup \gamma)$, which is the cone of $\Gamma(f)$ over 0.

We can see from the definition that the support of a polynomial f does not tell the difference of coefficients of monomials in it. For example $f_1(x_0, x_1) = x_0^2 + x_1^3$

and $f_2 = 2x_0^2 + 4x_1^3$ both have support $\{(2,0),(0,3)\}$. When we take the Newton polyhedron, i.e, the convex hull of the support of f, we lose information about f, but many properties of the polynomial and the algebraic variety defined by it depend only on the Newton polyhedron of f.

In some sense, the behavior of a polynomial is determined by the powers of the monomials in it. For example, by the fundamental theorem of algebra, the number of roots of a polynomial $f \in \mathbb{C}[x]$ is determined by the monomial of highest degree. The multiplicity of 0 as a root is captured by the monomial of lowest degree. Thus, the difference between the highest and the lowest degrees measures the number of roots in $\mathbb{C}^*$. This difference is exactly encoded in the size of the Newton polyhedron of f, which is the length of a line segment in the 1-dimensional case. This behavior generalizes to higher dimensions: the number of nonzero solutions to a generic system of polynomial equations equals the mixed volume of the Newton polyhedra of these polynomials.

Example 3.2. Take the polynomial with two variables $f = 2x_0^3 + 3x_0x_1 + x_1^4$. The support of f is given by $\operatorname{supp} f := \{(3,0),(1,1),(0,4)\}$. Then, the Newton polyhedron $\Gamma_+(f)$ is the convex hull of $\{m + x \mid m \in \operatorname{supp} f, x \in \mathbb{R}_+^n\}$.

The Newton boundary $\Gamma(f)$ consists of the line segment from $(3,0)$ to $(1,1)$ and the line segment from $(1,1)$ to $(0,4)$. The compact polyhedron $\Gamma_-(f)$ is the polygon with vertices $(0,0),(3,0),(1,1),(0,4)$.

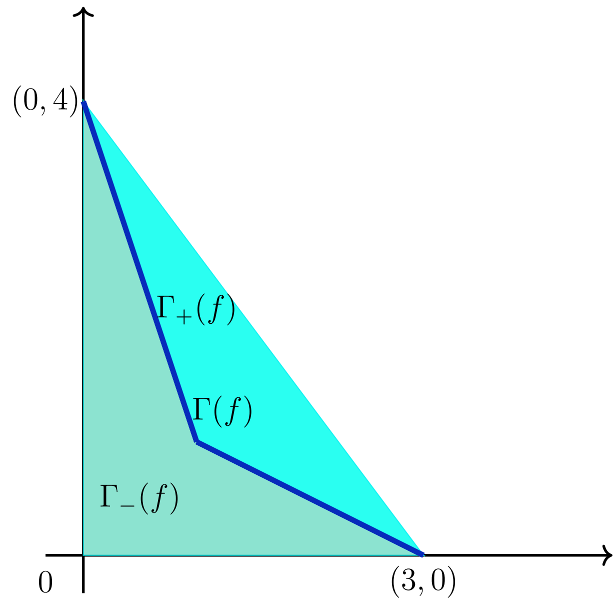

Figure 1: Newton polyhedron of $f = 2x_0^3 + 3x_0x_1 + x_1^4$

Definition 3.3. We call a polynomial f *convenient* if the Newton polyhedron of f intersects all coordinates, that is, for all $1 \leq i \leq n$, there exists monomials $x_i^{k_i}$ in f for some $k_i \in \mathbb{N}$ with nonzero coefficients.

Remark 3.4. This convenient condition guarantees that the polynomial is well-behaved. In particular, the partial derivatives vanish only in a certain range.

Definition 3.5. Let S be a compact polyhedron, then the *Newton number of the polytohedron* S is defined as

$$\nu(S) = n!V_n - (n-1)!V_{n-1} + \cdots + (-1)^{n-1}V_1 + (-1)^n,$$

where the V_n is the n-dimensional volume of S for $1 \leq k \leq n-1$ and V_k is the sum of the k-dimensional volumes of $S \cap \mathbb{R}^I$ where $I = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ and $\mathbb{R}^I = \{x = (x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ and } x_i = 0 \text{ if } i \notin I\}$.

Definition 3.6. When f is convenient, the *Newton number of* f, denoted by $\nu(f)$, is defined as the $\nu(\Gamma_-(f))$.

Example 3.7. Continuing from Example 3.2, we take $f = 2x_0^3 + 3x_0x_1 + x_1^4$. We may see that the Newton number of f is given by $\nu(f) = 2!V_2 - (2-1)!V_1 + (-1)^2 = 7 - 7 + 1 = 1$.

Definition 3.8. For a face γ of $\Gamma(f)$, we write $f_\gamma(z) := \sum_{m\in\gamma} a_m x^m$. We say that f is non-degenerate on $\Gamma(f)$ if for any face γ of $\Gamma(f)$, the equations $x_1 \frac{\partial f_\gamma}{\partial x_1}, \cdots, x_n \frac{\partial f_\gamma}{\partial x_n}$ have no common roots in $(\mathbb{C}^*)^n$.

For polynomials that are non-degenerate, solutions to $x_1 \frac{\partial f_\gamma}{\partial x_1} = \cdots = x_n \frac{\partial f_\gamma}{\partial x_n}$ all show up at points where at least one coordinate is 0. So, all solutions can be captured by looking at $\mathbb{C}^n \setminus (\mathbb{C}^*)^n$. Note that this condition is almost always satisfied: when we restrict to a face of the Newton boundary $\Gamma(f)$, there are fewer variables showing up, but since we still have n equations, there are generically no common solutions.

Now we will try to motivate the alternating sum formula for the Newton number $\nu(f)$ and its relation to the Milnor number $\mu_o(f)$. We start with the following lemma.

Lemma 3.9 ([Kou76]). *Let $g', g'', g_2, \ldots, g_n \in \mathbb{C}[x_1, \ldots, x_n]$ and let $g_1 = g'g''$. Then,*

$$\begin{aligned}\dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\ldots,x_n]_{\mathfrak{m}_0}/(g_1,g_2,\ldots,g_n)\right) =& \dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\ldots,x_n]_{\mathfrak{m}_0}/(g',g_2,\ldots,g_n)\right)\\ &+ \dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\ldots,x_n]_{\mathfrak{m}_0}/(g'',g_2,\ldots,g_n)\right).\end{aligned}$$

We may see that $V(g_1) = V(g'g'') = V(g') \cup V(g'')$. Since the dimension of the local algebra picks up the intersection multiplicity of at the point, this is just counting the total intersection multiplicity by summing up the ones contributed by $V(g')$ and $V(g'')$ respectively.

So, $\dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\ldots,x_n]_{\mathfrak{m}_0}/\left(x_1\frac{\partial f_\gamma}{\partial x_1},\cdots,x_n\frac{\partial f_\gamma}{\partial x_n}\right)\right)$ splits up into the sum of 2^n terms by iterative application of the lemma above, where one of these terms exactly gives the Milnor number at 0: $\mu_0(f)=\dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\ldots,x_n]_{\mathfrak{m}_0}/\left(\frac{\partial f_\gamma}{\partial x_1},\cdots,\frac{\partial f_\gamma}{\partial x_n}\right)\right)$.

Kouchnirenko [Kou76] discovered a relationship between the Newton polyhedron of f and $\dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\ldots,x_n]_{\mathfrak{m}_0}/\left(x_1\frac{\partial f_\gamma}{\partial x_1},\cdots,x_n\frac{\partial f_\gamma}{\partial x_n}\right)\right)$ if f is convenient and non-degenerate on the Newton boundary.

Lemma 3.10. *If $f\in\mathbb{C}[x_1,\ldots,x_n]$ is a convenient polynomial that is non-degenerate on the Newton boundary $\Gamma(f)$, then*

$$\dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\ldots,x_n]_{\mathfrak{m}_0}/\left(x_1\frac{\partial f}{\partial x_1},\cdots,x_n\frac{\partial f}{\partial x_n}\right)\right)=n!V_n,$$

where V_n is the n-dimensional volume of $\Gamma_-(f)$.

Since restrictions of Newton boundaries of convenient polynomials non-degenerate on $\Gamma(f)$ to hyperplanes $x_i=0$ give rise to faces of Newton boundaries, they still correspond to convenient polynomials non-degenerate on their Newton boundaries. The formula in the lemma above applies iteratively. For example,

$$\begin{aligned}&\dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\ldots,x_n]_{\mathfrak{m}_0}/\left(x_1,x_2\frac{\partial f_\gamma}{\partial x_2},\cdots,x_n\frac{\partial f_\gamma}{\partial x_n}\right)\right)\\ =&\dim_{\mathbb{C}}\left(\mathbb{C}[x_2,\ldots,x_n]_{\mathfrak{m}_0}/\left(x_2\frac{\partial f_\gamma}{\partial x_2},\cdots,x_n\frac{\partial f_\gamma}{\partial x_n}\right)\right)\\ =&(n-1)!\,\mathrm{volume}(\Gamma_-(f)\cap V(x_1))\end{aligned}$$

Now we may compute the Milnor number at 0, $\mu_0(f)$, by subtracting off the $2^n - 1$ unwanted terms. The unwanted terms have at least one component given by $x_i = 0$ for some $1 \leq i \leq n$. For each i, there are 2^{n-1} terms in total that have the i-th component given by $x_i = 0$. We obtain the sum of intersection multiplicities contributed by terms with $x_i = 0$: $\sum_{i=1}^{n}(n-1)!\,\mathrm{volume}(\Gamma_-(f) \cap V(x_i)) = (n-1)!V_{n-1}$. However, since we have double-counted the terms with $x_i = x_j = 0$ for all $i \neq j$, we need to add back them back. The sum of intersection multiplicities contributed by these terms equals $\sum_{i \neq j}(n-2)!\,\mathrm{volume}(\Gamma_-(f) \cap V(x_i, x_j)) = (n-2)!V_{n-2}$. But in doing so, we have added back the terms with $x_i = x_j = x_k = 0$ for some distinct i, j, k, which we do not want. To correct this, we can use the inclusion-exclusion principle and continue the same process recursively. Finally, we obtain the expression

$$\mu_0(f) = n!V_n - (n-1)!V_{n-1} + \cdots + (-1)^{n-1}V_1 + (-1)^n = \nu(f).$$

If f is degenerate on the Newton boundary $\Gamma(f)$, meaning that $x_1\frac{\partial f_\gamma}{\partial x_1}, \cdots, x_n\frac{\partial f_\gamma}{\partial x_n}$ do have common roots in $(\mathbb{C}^*)^n$ for some face γ, then the Newton number only serves as a lower bound.

Theorem 3.11 ([Kou76], Theorem 1)**.** *Let f be a convenient complex polynomial with an isolated singular point at 0 for $V(f)$. Then,*

- $\mu_0(f) \geq \nu(f)$,
- $\mu_0(f) = \nu(f)$ *if f is nondegenerate on* $\Gamma(f)$.

This result gives an lower bound of the Milnor number at the isolated singularity 0 by the Newton number, which depends only on the Newton polyhedron of f. Note that even degenerate polynomials on the Newton boundary can have the same value for $\mu_0(f)$ as for $\nu(f)$, as in the case of $f = (x_1 + x_2)^2 + x_1x_3 + x_3^2$.

Example 3.12. We may check the Milnor number of $f = 2x_0^3 + 3x_0x_1 + x_1^4$ at 0.

$$
\begin{aligned}
&\dim_{\mathbb{C}}\left(\mathbb{C}[x_0, x_1]_{\mathfrak{m}_0} / \left(\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}\right)\right) \\
=&\dim_{\mathbb{C}}\left(\mathbb{C}[x_0, x_1]_{\mathfrak{m}_0} / \left(6x_0^2 + 3x_1, 3x_0 + 4x_1^3\right)\right) \\
=&\dim_{\mathbb{C}}\left(\mathbb{C}[x_0, x_1]_{\mathfrak{m}_0} / \left(6x_0^2 + 3x_1, 3x_0 + 4x_1^3, x_1(8x_0x_1^2 - 3)\right)\right) \\
=&1.
\end{aligned}
$$

In this case, it is indeed true that $\mu_0(f) = \nu(f)$.

One of the assumptions for this theorem is that f needs to be convenient. However, a later lemma established by Oka [Oka79] demonstrated that this assumption was unnecessary. Oka's lemma is based on an important observation that we can transform any polynomial with isolated singularity into a convenient one while preserving the Milnor fibration:

Lemma 3.13. *Let f be an analytic function which has an isolated critical point at the origin. Consider a family $F(z, t) = f(z) + tz^A$ where $A = (A_1, \ldots, A^n) \in \mathbb{N}^n$ satisfies the condition that $|A| = |A_1| + \ldots + |A_n| \geq \mu(f, 0) + 2$. Then, the Milnor fibrations of $F(z, 0)$ and $F(z, 1)$ are equivalent.*

Example 3.14. For the polynomial $f = z^m + w^n$, we have that $\mathrm{supp}(f) = \{(m, 0), (0, n)\}$

and $\Gamma_-(f) = \mathrm{conv}(\{(0,0),(m,0),(0,n)\})$. The Newton boundary $\Gamma(f)$ is given by the line segment between $(m,0)$ and $(0,n)$.

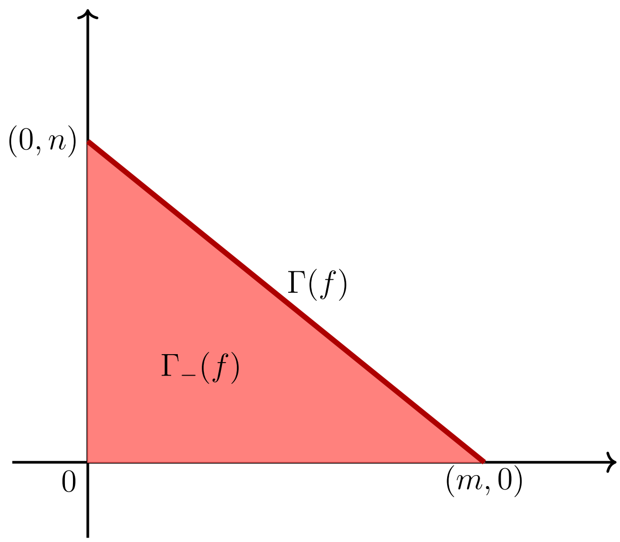

Figure 2: Newton polyhedron of $f = z^m + w^n$

The Newton number of this polynomial is

$$\nu(f) = 2!V_2 - (2-1)!V_1 + (-1)^2 = mn - (m+n) + 1 = (m-1)(n-1).$$

In this case, the Newton number of f exactly agrees with $\mu_0(f) = (m-1)(n-1)$.

One may check that the f is non-degenerate on $\Gamma(f)$: For the only face $\gamma \in \Gamma(f)$, $f_\gamma(z,w) = z^m + w^n$. And the equations $z\frac{\partial f_\gamma}{\partial z} = mz^m, w\frac{\partial f_\gamma}{\partial w} = nw^n$ indeed have no common roots in $(\mathbb{C}^*)^2$.

4 Milnor numbers under deformations

4.1 Milnor numbers and local topologies

Recall Lemma 2.8 from the previous section. If p is an isolated singularity, $S_\varepsilon \cap V(f)$ does not depend on ε for ε small enough. Moreover, for any ε and ε' that are sufficiently small, there always exists a diffeomorphism from S_ε to $S_{\varepsilon'}$ that brings K_ε to $K_{\varepsilon'}$ [Mil68]. Therefore, $(B_\varepsilon, B_\varepsilon \cap V(f))$ is homeomorphic to $(C(S_\varepsilon), C(S_\varepsilon \cap V(f)))$, where $C(X)$ denotes the cone over X. It is worth noting that the space K_ε alone does not fully determine the local topology of an isolated singularity. In fact, for any coprime pair m, n, the links defined by $f = z^m + w^n \in \mathbb{C}[z, w]$ are torus knots and thus are homeomorphic to each other, yet the local topologies at the singularity point 0 are different. To understand the local topology, it is essential to study how K_ε embeds in S_ε, or equivalently, how $B_\varepsilon \cap V(f)$ embeds in B_ε.

Now consider another complex polynomial g with an isolated singularity at the same point p. A natural question to ask is whether f and g have the same topological type at p, that is, whether there exists a homeomorphism

$$(B_\varepsilon, B_\varepsilon \cap V(f)) \to (B_\varepsilon, B_\varepsilon \cap V(g)).$$

This question is closely related to the Milnor numbers of the isolated singularity point.

Let us first note that the Milnor number is an invariant of the hypersurface. That is, if there is another polynomial g that defines the same hypersurface and shares the same singularity with f, then the two Milnor numbers are the same.

Lemma 4.1 ([CLS20]). *Suppose polynomials $f, g \in \mathbb{C}[x_0, \ldots, x_n]$ define the same hypersurface $V(f) = V(g)$, with the same isolated singularity at p. Then, $\mu_p(f) = \mu_p(g)$.*

We can see this using the interpretation of the Milnor number as the dimension of the local algebra.

When f and g define the same hypersurface with the same isolated singularity, we have a one-parameter family of functions $F(x,t) = (1-t)f(x) + tg(x)$ interpolating between f and g.

As t varies, the dimension of the local algebra is constant by the preservation of the intersection multiplicity.

In this case, since f and g define the same hypersurface, they have the same topological type, given by the identity map $(B_\varepsilon, B_\varepsilon \cap V(f)) \to (B_\varepsilon, B_\varepsilon \cap V(g))$.

However, the condition $V(f) = V(g)$ is not strictly necessary. Teissier's theorem [Tei73] established that the Milnor number is an invariant under under topological types.

Theorem 4.2. *If there is a homeomorphism $(B_\varepsilon, B_\varepsilon \cap V(f)) \to (B_\varepsilon, B_\varepsilon \cap V(g))$, then $\mu_p(f) = \mu_p(g)$.*

Proof. Since $(B_\varepsilon, B_\varepsilon \cap V(f)) \cong (B_\varepsilon, B_\varepsilon \cap V(g))$, the complement of $V(f)$ in B_ε is homeomorphic to the complement of $V(g)$ in B_ε. By Theorem 2.7 (Milnor's fibration theorem, second version), we know that these two sets are trivial smooth fiber bundles over a punctured disc. Thus, the fibers must be homotopy equivalent. This implies that $\mu_p(f) = \mu_p(g)$. $\square$

Hence, if we have a family $F(x, t)$ of hypersurfaces all with the same topological type at singularities, then their Milnor numbers will be the same.

Lê and Ramanujam gave the proof for the converse of this statement when the dimension of the hypersurface is not 2:

Theorem 4.3 ([TR76])**.** *Let $F(t, x)$ be a polynomial in $x = (x_0, \ldots, x_n)$ with coefficients which are smooth complex valued functions of $t \in I = [0, 1]$ such that for each $t \in I$, $F(t, 0) = 0$ and $\frac{\partial F}{\partial x_i}(t, x)$ in x has an isolated zero at 0. Assume moreover that the integer $\mu_t = \dim_{\mathbb{C}} \left(\mathbb{C}\{x\} / \left(\frac{\partial f}{\partial x_0}(t, x), \ldots, \frac{\partial f}{\partial x_n}(t, x)\right)\right)$ is independent of t, where $\mathbb{C}\{x\}$ is the ring of convergent power series. Then the monodromy fibrations of the singularities of $F(0, x) = 0$ and $F(1, x) = 0$ at 0 are of the same fiber homotopy. Further, if $n \neq 2$, these fibrations are differentiably isomorphic and the topological types of the singularities are the same.*

The above result implies that if we have a smooth family of singular hypersurfaces, with the same isolated singular point p and Milnor numbers at p, then these hypersurfaces have the same local topological type at p. This statement holds true for all complex hypersurfaces with dimension $n \neq 2$. For $n = 1$, the proof is relatively simple, and for higher dimensions, the proof involves the h-cobordism theorem which requires the assumption that $n \geq 3$. As of now, the question is still open for dimension 2.

The $n = 2$ case has been partially answered positively in a later paper by Parusiński [Par99] with additional assumption that the family of hypersurfaces given is of the form $F(x,t) = f(x) + tg(x)$ where $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

Proposition 4.4. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be such that $F(x,t) = f(x) + tg(x)$ give a family of hyperspaces with isolated singularities with constant Milnor numbers. Then F is topologically trivial.*

An answer [Oka79; Abd16] has also been given for another specific family of hypersurfaces, extending the work by Kouchnirenko on Newton polyhedra.

It follows from 3.11 that if a holomorphic function f with an isolated singularity at 0 is non-degenerate on the Newton boundary $\Gamma(f)$, then the Milnor number at 0 equals the Newton number $\nu(f)$, which is completely determined by $\Gamma(f)$.

Oka [Oka79] proved that, more than just the Milnor numbers, the Milnor fibration, and thus the topological type of the singularity, is determined by the Newton boundary as well.

Theorem 4.5. *Suppose $F(x,t)$ gives a non-degenerate family of hypersurfaces with an isolated singularity at 0, and the Newton boundary of $F(x,t)$ is independent of t. Then the Milnor fibration of $F(x,t)$ is independent of t.*

In the case that the non-degenerate family of hypersurfaces have the same Newton boundary, the local topology is invariant at the common isolated singularity point.

Abderrahmane [Abd16] generalized this result for a family of hypersurfaces that are non-degenerate on the Newton boundary.

Theorem 4.6. *Let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a one parameter deformation of a holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity such that the Milnor number $\mu(F_t)$ is constant. Suppose that F_t is non-degenerate for every t. Then F_t is topologically trivial.*

Remark 4.7. In Abderrahmane's result, this family of hypersurfaces is not required to have the same Newton boundary. The assumption of constant Milnor number is sufficient to guarantee that all the fibers have the same topological type at the isolated singularity.

4.2 Connections to the multiplicity of polynomials

The local topology of the singularity p is closely related to the multiplicity of f, which is expressed as the lowest degree in the power series expansion of f at $p \in \mathbb{C}^n$. We have shown above how the constancy of the Milnor numbers leads to the invariance of the topological type. A similar question has been suggested by Zariski [Zar65] along this line: what does the constancy of the Milnor numbers tell us about the multiplicity of the functions defining these hypersurfaces at the fixed isolated singular point?

Recall that the constancy of Milnor numbers implies the invariance of the topological type for hypersurfaces of dimension other than two [TR76]. For plane curve in particular, the multiplicity of plane curves is a topological invariant and thus the constant Milnor numbers implies constant multiplicity as well. The question has not been fully answered in other dimensions. But with additional hypotheses on the functions, we have obtained the following results: Gert-Martin Greuel [Gre86] generalized earlier works on homogeneous hypersurfaces [GK75] and illustrated that if we only consider weighted homogeneous polynomials, then the constancy of Milnor numbers implies the constancy of multiplicities. Combined with Teissier's result [Tei73], this implies that the multiplicity is an invariant of the topological type for weighted homogeneous polynomials.

Oka [Oka89] showed that a non-degenerate family $F(x,t) = f(x) + tz^A$ for some fixed monomial $z^A = z_1^{A_1} \dots z_n^{A_n}$ with invariant Milnor number is topologically trivial. Generalizing Oka's result, Abderrahmane [Abd16] proved the following theorem:

Theorem 4.8. *Let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a one parameter deformation of a holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity such that the Milnor number $\mu(F_t)$ is constant. Suppose that F_t is non-degenerate for every t. Then, the multiplicity of functions F_t is constant.*

5 Global Milnor numbers

In this section, we consider a specific family of hypersurfaces $\{f^{-1}(t) : t \in \mathbb{C}\}$ and discuss the the connections between this family of hypersurfaces and the Milnor numbers.

Recall that the fiber F_θ of the Milnor fibration is diffeomorphic to $f^{-1}(t) \cap B_\varepsilon$ for an epsilon ball centered at a singularity point p, for $t \in \mathbb{C}$ sufficiently close to 0. Hence, the topology of hypersurfaces $f^{-1}(t)$ for t close to 0 depends on the local topology of the singularity point on the hypersurface $f^{-1}(0)$.

Now consider a family of hypersurfaces $\{f^{-1}(t) : t \in \mathbb{C}\}$. We see that singular points showing up on this family of hypersurfaces exactly come from solutions to the equation $\nabla f = 0$. We refer to the set $V(\nabla f)$ as the *critical set* of f. Then an interesting question to ask is how the critical set $V(\nabla f)$ determines the topology of the family $\{f^{-1}(t) : t \in \mathbb{C}\}$.

We call $t \in \mathbb{C}$ a *critical value* of f if there exists a critical point in $f^{-1}(t)$. Note that the critical set $V(\nabla f) = V(\nabla f)$ is an algebraic set and hence a finite union of irreducible algebraic varieties. Since ∇f is 0 on $V(\nabla f)$ and any irreducible algebraic variety is connected, the function value of f on each irreducible algebraic variety in $V(\nabla f)$ must be constant. Therefore, the set of critical values is finite. In other words, only a finite subcollection of the family $\{f^{-1}(t) : t \in \mathbb{C}\}$ are singular.

In particular, when f only has isolated critical points, $V(\nabla f)$ is finite. We take the sum of the Milnor numbers for isolated singular points in this family $\{f^{-1}(t) : t \in \mathbb{C}\}$, denoted by $\widetilde{\mu}(f)$. Algebraically,

$$\widetilde{\mu}(f) = \dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\dots,x_n]\Big/\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right)\right).$$

Note that if $V(\nabla f)$ is a finite set of discrete points,

$$\widetilde{\mu}(f) = \dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\dots,x_n]\Big/(\nabla f)\right) = \sum_{p\in V(\nabla f)} \dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\dots,x_n]\Big/(\nabla f)\right)_{\mathfrak{m}_p}.$$

Recall from Section 3, Kouchnirenko [Kou76] gave a lower bound of the Milnor number at an isolated singularity by the Newton number. Similarly, he gave an upper bound of the $\widetilde{\mu}(f)$, for convenient polynomials with only isolated singularities.

Definition 5.1. Let $\widetilde{\Gamma}_-(f) := \mathrm{conv}(\{0\}\cup \mathrm{supp}(f))$ Define *Newton boundary of the polynomial f at infinity* as the union of all faces of $\widetilde{\Gamma}_-(f)$ that do not contain 0, denoted by $\widetilde{\Gamma}(f)$. The *Newton number at infinity of f* is defined as $\widetilde{\nu}(f) := \nu(\widetilde{\Gamma}_-(f))$.

Definition 5.2. Similar to before, we say that f is *non-degenerate on* $\widetilde{\Gamma}(f)$ if for any face γ of $\widetilde{\Gamma}(f)$, the equations $x_0\frac{\partial f_\gamma}{\partial x_0},\cdots,x_n\frac{\partial f_\gamma}{\partial x_n}$ have no common roots in $(\mathbb{C}^*)^n$.

Theorem 5.3 ([Kou76])**.** *Let f be a convenient function.*

- $\widetilde{\mu}(f) \le \widetilde{\nu}(f)$,
- *If f is non-degenerate on $\widetilde{\Gamma}(f)$, then $\widetilde{\mu}(f) = \widetilde{\nu}(f)$.*

Remark 5.4. Unlike the local version stated in Theorem 3.11, Oka's approach cannot be extended to arbitrary polynomials in the global case. This is due to the fact that adding a monomial z^A of sufficiently high degree to the polynomial may alter the Newton boundary at infinity $\widetilde{\Gamma}(f)$.

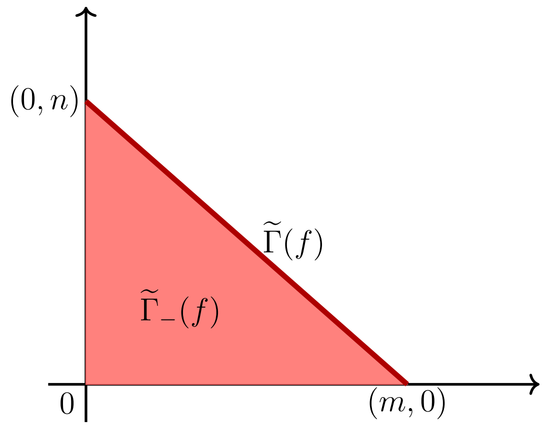

Figure 3: Newton Boundary of $f = z^m + w^n$ at infinity

Remark 5.5. In the variety of polynomials with a given Newton Boundary at infinity, the non-degenerate polynomials form an open dense subvariety [Bro88].

Example 5.6. For the example $f = z^m + w^n$, $\widetilde{\Gamma}_-(f) = \Gamma_-(f)$. The Newton number at infinity $\widetilde{\nu}(f) = \nu(f) = (m-1)(n-1)$ gives an upper bound for the sum of Milnor numbers $\mu(f)$. This indeed coincides with the observation that the set of critical points determined by f only consists of one point $(0,0)$.

However, the finiteness of the set of critical values does not imply that f induces a locally trivial smooth fibration over a neighborhood of a noncritical point. For example, the function $f = x - x^2y$ has no critical points, but there does not exist any neighborhood of $0 \in \mathbb{C}$ such that f restricts to a locally trivial fibration. We call $a \in \mathbb{C}$ an atypical value of f if there does not exist any neighborhood $U \subset \mathbb{C}$ of a such that f induces a locally trivial smooth fibration $f^{-1}(U) \to U$. Denote the set of atypical values of f by $B(f)$, which is often called the bifurcation set in the literature. Over the complement of $B(f)$ in $\mathbb{C}$, the map $f\colon \mathbb{C}^n \to \mathbb{C} \setminus B(f)$ is a locally trivial fibration [Tho69]. We call $f^{-1}(t)$ a generic fiber for $t \in \mathbb{C} \setminus B(f)$. One may see that $B(f)$ contains the set of critical values as a finite subset. It has been shown that the set of atypical values is actually finite as well [Ver76]. The atypical values that are not critical values appear because there are "critical points at infinity" associated to $f^{-1}(t)$ [Bro88]. Broughton [Bro88] introduced a class of polynomials known as tame polynomials, which limits the polynomials to those with nice behavior at infinity. Then, he showed that the topology of a fiber $f^{-1}(t)$ relates to the Milnor numbers.

Definition 5.7. Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial. We call f a *tame* polynomial if there exists a compact neighborhood N of all the critical points of f such that $\|\nabla f(x)\|$ is bounded away from 0 for all $x \in \mathbb{C}^n \setminus N$.

Tame polynomials may be characterized in terms of Milnor numbers:

Proposition 5.8. *A polynomial f is tame if and only if $\widetilde{\mu}(f)$ is finite and $\widetilde{\mu}(f) = \widetilde{\mu}(f^w)$ for all sufficiently small $w \in \mathbb{C}^n$, where*

$$f^w(x_1, \ldots x_n) := f(x_1, \ldots, x_n) - (w_1 x_1 + \ldots + w_n x_n).$$

Remark 5.9. The concept of "tame polynomials" generalizes convenient polynomials with nondegenerate Newton Boundary at infinity in Kouchnirenko's theorems - these polynomials are automatically tame. This follows directly from the proposition above. If we perturb a convenient polynomial by a linear term, then the polynomial still have the same Newton Boundary at infinity, unless the coefficients exactly cancel out. In another words, if w is small enough, then $\widetilde{\Gamma}(f) = \widetilde{\Gamma}(f^w)$ and thus $\widetilde{\mu}(f) = \widetilde{\mu}(f^w)$.

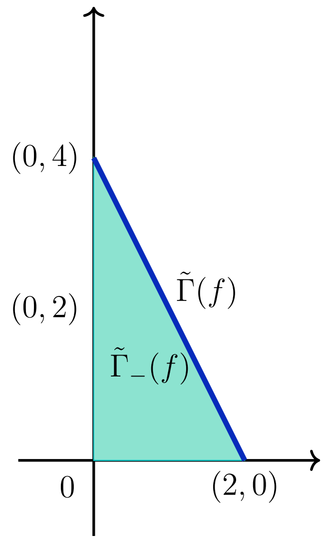

Figure 4: Newton Boundary of $f = x_1^2 + 2x_1x_2^2 + x_2^4 + x_2^2$ at infinity

Example 5.10. Let $f(x_1, x_2) = x_1^2+2x_1x_2^2+x_2^4+x_2^2 = (x_1+x_2^2)^2+x_2^2$. If $w = (a, b) \in \mathbb{C}^2$ and $a \neq -1$, then $f^w(x_1, x_2) = x_1^2+2x_1x_2^2+x_2^4+x_2^2-ax_1-bx_2$ has one non-degenerate critical point and $\widetilde{\mu}(f^w) = 1$. So f is tame. However, $\widetilde{\nu}(f) = 8-(4+2)+1 = 3 > \widetilde{\mu}(f)$, so f must be degenerate on the Newton Boundary at infinity.

For a tame polynomial, Broughton[Bro88] established the relationship between fibers $f^{-1}(t)$ and Milnor numbers:

Theorem 5.11. *Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a tame polynomial. Let $\widetilde{\mu}, \widetilde{\mu}^t$ be the sum of Milnor numbers of f and the sum of Milnor numbers on $f^{-1}(t)$ respectively for $t \in \mathbb{C}$. Then for any $t \in \mathbb{C}$, the corresponding fiber $f^{-1}(t)$ has the homotopy type of a bouquet of $\widetilde{\mu} - \widetilde{\mu}^t$ spheres of dimension $n-1$. In particular, for a generic fiber $f^{-1}(t)$ where $t \in \mathbb{C} \setminus B(f)$, it has the homotopy type of a bouquet of $\widetilde{\mu}$ spheres of dimension $n-1$.*

We may also see that the theorem above implies that if $f\colon \mathbb{C}^n \to \mathbb{C}$ is a convenient polynomial non-degenerate on $\widetilde{\Gamma}(f)$, then a generic fiber $f^{-1}(t)$ has the homotopy type of a bouquet of $\widetilde{\nu}(f)$ spheres of dimension $n-1$.

Bartolo et al. [ALM00] extended this result to non-convenient polynomials and established a relationship between the Euler characteristic of a generic fiber $f^{-1}(t)$ and the Newton number at infinity $\widetilde{\nu}(f)$ of f. Specifically, they proved the following theorem.

Theorem 5.12. *Let f be a complex polynomial. Let $t \in \mathbb{C} \setminus B(f)$. If f is non-degenerate on $\widetilde{\Gamma}(f)$, then*

$$(-1)^{n-1}(\chi(f^{-1}(t)) - 1) = \widetilde{\nu}(f).$$

6 A generalization to non-isolated singularities

In the previous sections, most of the results and characterizations are restricted to when the singularity is isolated. We might ask what should replace the Milnor number for a non-isolated hypersurface singularity.

Adam Parusiński [Par88] gave a generalization of the Milnor number to a compact component of the set of singular points of a hypersurface of a complex manifold.

He started out with the following observation:

Proposition 6.1 ([Par88]). *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ with an isolated singular point of $V(f)$. Then the local Poincaré-Hopf index of the vector field $(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})$ at p over $\mathbb{C}^{n+1}$ equals the Milnor number of $V(f)$ at p.*

Since $\nabla f = (\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defines a section of a trivial vector bundle $T\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, the local Poincaré-Hopf index at a point agrees with the local topological degree, which is the same as the Milnor number at a point.

Now consider a n-dimensional connected complex manifold M. Let X be a hypersurface in M, i.e., the vanishing locus of a holomorphic section v of a holomorphic line bundle L over M. If we fix a Hermitian metric on L, we can decompose the associated connection D into $D = D' + D''$, where

$$D'\colon \mathscr{A}^0(L) \to \mathscr{A}^{1,0}(L) = \mathscr{A}^0\left(T^{*\prime}M \otimes L\right),$$

$$D'' = \bar{\partial}\colon \mathscr{A}^0(L) \to \mathscr{A}^{0,1}(L).$$

The bundle $\mathscr{A}^{1,0}(L)$ consists of $(1,0)$-forms on M with values in L, which are given by sections of $T^{*\prime}M \otimes L$, where $T^{*\prime}M$ is the holomorphic cotangent bundle of M. The bundle $\mathscr{A}^{0,1}(L)$ consists of $(0,1)$-forms on M with values on L.

Any holomorphic section v is given by holomorphic functions in local holomorphic trivializations. As a consequence, v must be annhilated by the operator $D'' = \bar{\partial}$. In this way, we obtain an expression for the set of singular points of the hypersurface X:

$$\mathrm{Sing}(X) := \{x \in X : D'v = 0\}$$

Now suppose $\mathrm{Sing}(X)$ is more than a discrete set of points. Since $\mathrm{Sing}(X)$ is closed and open in the zero set of $D'v$ [Par88], $\mathrm{Sing}(X)$ consists of smooth compact connected components.

Let Y be a smooth compact m-dimensional component of $\mathrm{Sing}(X)$ and U be a small neighborhood of X. Parunsiński defined a notion of Milnor numbers at Y, using the intersection index:

Definition 6.2. The *generalized Milnor number of X at Y*, denoted by $\mu(X;Y)$, is defined as the intersection index $\mathrm{ind}_U\, D'v$, where v is the section defining the hypersurface X.

Definition 6.3. Assume that X is compact. The *generalized Milnor number of X*, denoted by $\mu(X)$, is defined as the intersection index of the zero section of $T^{*\prime}M \otimes L$ and $D'v$ over a small neighborhood of X where v is the section defining the hypersurface X.

Remark 6.4. When $Y = \{p\}$ is a singleton, $\mu(X;Y)$ is exactly the Milnor number of v at p defined before. If X is compact and only consists of isolated singular points, then the generalized Milnor number of X, $\mu(X)$, is just the sum of the Milnor numbers at singular points.

In this more general setting, $\mu(X) = \sum_{i=1}^{r} \mu(X, Y_i)$, where $Y_1, \ldots, Y_r$ are connected components of $\mathrm{Sing}(X)$.

To each point $x \in Y$, one can attach a sequence of *Teissier numbers*

$$\mu^{n-m}(X, x), \ldots, \mu^{1}(X, x)$$

The Teissier number $\mu^i(X, x)$ is defined as the Milnor number of $X \cap H$ at x where H is a generic i-dimensional hyperplane of $\mathbb{C}^n$ that passes through p [Tei73]. Teissier showed that the pair $(X \setminus \mathrm{Sing}(X), Y)$ satisfies Whitney's conditions if and only if this sequence of Teissier numbers is constant on Y [Tei82]. We shall not explain all the details involving Whitney stratification here, but the intuition is that a Whitney stratification allows us to decompose a singular hypersurface into a disjoint union of smooth manifolds in a compatible way. When the sequence of Teissier numbers is constant on Y, Parusiński gave an explicit computation of the Milnor number using Chern classes:

Proposition 6.5. *If the pair $(X \setminus \mathrm{Sing}(X), Y)$ satisfies Whitney's conditions, then*

$$\mu(X, Y) = \mu^{n-m}(X, y) \cdot \langle c_m(T^{*\prime}Y \otimes L|_Y), [Y] \rangle$$

where $[Y]$ is the fundamental homology class of Y and $c_i(E)$ denotes the i-th Chern class of Y.

Assuming M is compact, Parusiński [Par88] showed that the Milnor numbers are related to Euler characteristics by the following formula:

Proposition 6.6. $\mu(X) = (-1)^n\chi(X) + \langle c_n(T^{*\prime}M \otimes L), [M]\rangle - (-1)^n\chi(M)$ *where* $[M]$ *denotes the fundamental homology class of* M*.*

Suppose we have another hypersurface of M that is linearly equivalent to X, denoted X', this means that X' could be written as the vanishing locus of sections of the same line bundle L. Then it follows from the proposition above that

$$\mu(X') = (-1)^n\chi(X') + \langle c_n(T^{*\prime}M \otimes L), [M]\rangle - (-1)^n\chi(M).$$

$$\mu(X) - \mu(X') = (-1)^n(\chi(X) - \chi(X')).$$

Corollary 6.7. *If* X, X' *are equivalent as divisors, then*

$$\mu(X) - \mu(X') = (-1)^n(\chi(X) - \chi(X')).$$

This corollary states that if we have a smooth hypersurface X' linearly equivalent to X in a compact manifold M, then the difference between the Euler characteristic of X' and that of X is exactly measured by the sum of Milnor numbers of X at all singular points. To put it in another words, if we approximate the singular hypersurface X by a linear family of smooth hypersurfaces given by sections of the line bundle L, then $\mu(X)$ equals the change in the Euler characteristic as this family degenerates into X.

6.1 Hypersurfaces of $\mathbb{CP}^n$

In this subsection, we take consider the n-dimensional compact manifold $\mathbb{CP}^n$ and apply some of the Pausiński's results to consider hypersurfaces in this case. A complex projective hypersurface is given by the vanishing locus of a homogeneous polynomial. Since the diffeomorphism type of a smooth complex projective hypersurface is determined uniquely by its degree and dimension, let's assume $V(F)$ is given by $F = x_0^d + \ldots + x_n^d$. Then consider

$$F_a\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1};$$

$$(x_0, \ldots, x_n) \mapsto x_0^d + \ldots + x_n^d.$$

The affine cone on $V(F)$ in $\mathbb{C}^{n+1}$ is a complex hypersurface

$$V(F_a) = \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} \mid x_0^d + \ldots + x_n^d = 0\}$$

in $\mathbb{C}^{n+1}$ with an isolated singularity at 0, since $V(F_a, \nabla F_a) = \{0\}$.

For a hypersurface $V(g)$ in $\mathbb{C}^{n+1}$ given by a homogeneous polynomial $g\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, there exists a *global Milnor fibration* $g^{-1}(1) \hookrightarrow \mathbb{C}^{n+1} \setminus V(g) \to \mathbb{C}^*$, such that the fiber $g^{-1}(1)$ of this map is homotopy equivalent to the fiber of the Milnor fibration of f at 0 [Mil68].

Thus we have a global Milnor fibration:

$$F_a^{-1}(1) \hookrightarrow \mathbb{C}^{n+1} \setminus V(F_a) \to \mathbb{C}^*.$$

By Theorem 2.9, $F_a^{-1}(1)$ is homotopy equivalent to a finite bouquet of n-dimensional spheres, where the number of spheres is given by the Milnor number of F_a at 0:

$$\mu_p(F_a) = \dim_{\mathbb{C}} \left(\mathbb{C}\left[x_0, \ldots, x_n\right] \Big/ \left(\frac{\partial F_a}{\partial x_0}, \ldots, \frac{\partial F_a}{\partial x_n} \right) \right)_{\mathfrak{m}_0} = (d-1)^{n+1}.$$

This gives the Euler characteristic of the fiber:

$$\chi(F_a^{-1}(1)) = 1 + (-1)^n \mu_0(F_a)) = 1 + (-1)^n (d-1)^{n+1}.$$

Moreover, the map that goes from $F_a^{-1}(1)$ to $\mathbb{CP}^n \setminus X$ is a d-fold cover. So the Euler characteristic of $F_a^{-1}(1)$ relates to the Euler characteristic of $V(F)$ by

$$\chi(F_a^{-1}(1)) = d \cdot \chi(\mathbb{CP}^n \setminus V(F)) = d \cdot (\chi(\mathbb{CP}^n) - \chi(V(F)).$$

Thus, combining the two formulas above, we obtain the Euler characteristic of $V(f)$ in $\mathbb{CP}^n$:

$$\chi(F_a^{-1}(1)) = (n+1) - \frac{1}{d}\left(1 + (-1)^n (d-1)^{n+1}\right).$$

Now consider a singular hypersurface X in $\mathbb{CP}^n$ that is linearly equivalent to a degree-d smooth complex projective hypersurface in $\mathbb{CP}^n$. By Corollary 6.7, the Euler characteristic of X is given by

$$\chi(X) = (n+1) - \frac{1}{d}\left(1 + (-1)^n (d-1)^{n+1}\right) + (-1)^n \mu(X).$$

where $\mu(X)$ is the generalized Milnor number of X defined by Parusiński [Par88].

6.2 Milnor numbers at infinity

Going back to the case where $f \in \mathbb{C}[x_1, \ldots, x_n]$ defines a complex hypersurface in the affine space $\mathbb{C}^n$ with isolated singularities, we see that the homogenization of this polynomial in $\mathbb{CP}^n$ defines a hypersurface in $\mathbb{CP}^n$. Suppose $f = f_0 + f_1 + \ldots + f_d$ is a decomposition of this polynomial by degree. Then its homogenization F is given by $F = x_0^d f_0 + x_0^{d-1} f_1 + \ldots + f_d$.

Note that $V(f) \hookrightarrow V(F)$ along the natural inclusion $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$. In particular, isolated singularities of $V(f)$ will stay isolated in $V(F)$. Let D be the divisor of $\mathbb{CP}^{n-1}$ defined by the zero locus of the homogeneous polynomial f_d. This is the same as the restriction of the zero locus of F to the hyperplane at infinity $\mathbb{CP}^{n-1} = V(x_0)$.

The fact that the function f has only isolated singularity points in $\mathbb{C}^n$ allows us to separate the affine singular points from the singularities at infinity via the following equation [ALM00]:

$$\mu(\mathbb{CP}^n; V(F)) = \mu(\mathbb{CP}^n; V(F), D) + \mu(\mathbb{CP}^n; V(F), Sing(f)).$$

As a final remark, it is worth noting that while the polynomial f we started with had isolated singularities, the singularities that appear on the hyperplane at infinity do not have to be isolated.

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