LINEAR REPRESENTATIONS OF THE MAPPING CLASS GROUP OF DIMENSION AT MOST $3g - 3$

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Abstract. We classify representations of the mapping class group of a surface of genus g (with at most one puncture or boundary component) up to dimension $3g-3$. Any such representation is the direct sum of a representation in dimension $2g$ or $2g+1$ (given as the action on the (co)homology of the surface or its unit tangent bundle) with a trivial representation. As a corollary, any linear system on the moduli space of Riemann surfaces of genus g in this range is of algebro-geometric origin.

1. Introduction

Let S be an oriented surface of genus g, either closed, with one puncture, or with one boundary component. Assume $g \geqslant 3$ throughout. Let $\mathrm{Mod}(S)$ denote the mapping class group of S. This paper gives a classification of complex representations of $\mathrm{Mod}(S)$ in the dimension range $n \leqslant 3g-3$. In dimension $2g$, there is the the *symplectic representation*

$$\Psi : \mathrm{Mod}(S) \to \mathrm{GL}(H),$$

where $H = H_1(S;\mathbb{C})$ denotes the first homology of S, equipped with the intersection pairing $\langle \cdot, \cdot \rangle$. Moving up one dimension, there is a representation of dimension $2g+1$

$$\widetilde{\Psi} : \mathrm{Mod}(S) \to \mathrm{GL}(\widetilde{H}),$$

where $\widetilde{H} \cong \mathbb{C}^{2g+1}$ is a non-semisimple representation surjecting onto H given by the action on the homology of the unit tangent bundle of S; see Section 2.2. There is also the non-isomorphic dual representation $\widetilde{H}^*$.

Our main result shows that this gives a *complete* list of representations up to dimension $3g-3$.

Theorem A. *Let $g \geqslant 3$, and let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ be a nontrivial representation. Then for $n \leqslant 3g-3$, ρ is the direct sum of a trivial representation with one of the following:*

$$H, \widetilde{H}, \text{ or the dual } \widetilde{H}^*.$$

If S is closed, only H can appear.

This extends the work of Korkmaz [Kor23], who showed the uniqueness of the symplectic representation up to dimension $2g$, and Kasahara [Kas24], who classified representations of dimension $2g+1$.

Idea of proof. Our analysis is centered around the notion of a *bi-affine representation*, the basic theory of which (for arbitrary groups) is established in Section 3. A group representation V is bi-affine if there is a filtration $V_1 \leqslant V_2 \leqslant V$ for which V_1 and V/V_2 are both trivial. Building off of the ideas developed by Korkmaz and Kasahara, we show inductively that every representation of $\mathrm{Mod}(S)$ up to dimension $3g-3$ is bi-affine. To do this, we develop a criterion for a $\mathrm{Mod}(S)$-representation to be bi-affine in Proposition 5.4, and show inductively that any representation of dimension at

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most $3g-3$ satisfies the two conditions of Proposition 5.4 in Proposition 6.5 and Proposition 7.6. Separately, we show that any bi-affine representation of $\mathrm{Mod}(S)$ is in fact a direct sum of a trivial representation with a representation of dimension $2g+1$. This is a consequence of a cohomological calculation carried out in Proposition 4.6.

It is reasonable to ask why we use the language of bi-affine representations at all, if ultimately we are showing a stronger result. The answer is that we believe that this leads to the cleanest proofs of our results, with the least amount of fussing about cases and choosing coordinates. It is relatively painless to formulate a condition under which an extension of a bi-affine representation of $\mathrm{Mod}(S)$ remains bi-affine (cf. Lemma 4.2); the corresponding statement for an extension to be a direct sum with a trivial representation would be more elaborate, and would require us to carry around the data of the splitting for longer than necessary.

Interpretation in terms of local systems on moduli spaces. Local systems on the moduli space $\mathcal{M}$ of Riemann surfaces are determined by monodromy representations

$$\pi_1^{\mathrm{orb}}(\mathcal{M}) \to \mathrm{GL}(V),$$

where $\pi_1^{\mathrm{orb}}(\mathcal{M})$ is the mapping class group. A local system $\mathbb{V}$ on $\mathcal{M}$ is *geometrically constructible* if there is a family $F \to E \to \mathcal{M}$ of smooth projective varieties over $\mathcal{M}$ such that $\mathbb{V}$ is a subquotient of the local system of (co)homology associated with E. It is conjectured that all semi-simple representations of mapping class groups are geometrically constructible [Lit24].

Geometrically constructible local systems are necessarily semi-simple, while the local systems associated with $\widetilde{H}$ and $\widetilde{H}^*$ are not. For the purposes of this discussion, we will say that a local system *arises algebro-geometrically* if it is the monodromy of a family as above, where the fibers F are now only required to be quasiprojective. Theorem A imposes strong constraints on the local systems that can appear on moduli spaces of closed surfaces or of surfaces with a puncture – up to rank $3g-3$, they must arise algebro-geometrically.

Corollary B. *For $g \geq 3$, any local system of rank at most $3g-3$ on $\mathcal{M}_{g,1}$ or $\mathcal{M}_g$ arises algebro-geometrically. Here are the families of algebraic varieties:*

- *for the symplectic representation H, it is the universal family of Riemann surfaces;*
- *for the representations $\widetilde{H}$ and $\widetilde{H}^*$, it is the relative tangent bundle of the universal family of curves with the zero section removed - the fiber of this bundle is homotopy equivalent to the unit tangent bundle of surfaces.*

Applications to rigidity. In [Far24], Farb proves that any nonconstant holomorphic map $f : \mathcal{M}_{g,n} \to \mathcal{A}_g$ must be the period mapping assigning a Riemann surface to its Jacobian (here, $\mathcal{A}_g$ denotes the moduli space of principally polarized Abelian varieties of dimension g). The first step in the argument is to appeal to Korkmaz's work classifying representations of the mapping class group up to dimension $2g$, as this governs the possibilities for the induced map on orbifold fundamental groups. The work of this paper opens the way to extending Farb's work to give a classification of holomorphic maps $\mathcal{M}_g \to \mathcal{A}_h$ in the range $2h \leqslant 3g-3$. We plan to revisit this topic in future work.

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2. A recollection of prior results

2.1. **Cocycles and group cohomology.**
The construction and classification of bi-affine representations is formulated in the language of group cohomology. Here we recall the fragment of the theory that we need. For a proper introduction to the subject, see [Bro94].

For a group G and a left $\mathbb{C}[G]$-module M, a 1-*cocycle* (also called a *crossed homomorphism*) is a function $f : G \to M$ satisfying the equation

$$f(gh) = f(g) + gf(h)$$

for all $g, h \in G$. The group of 1-cocycles is denoted $Z^1(G, M)$. There is a *coboundary map*

$$\delta : M \to Z^1(G, M)$$

which sends $v \in M$ to the function

$$\delta(v)(g) = (g-1)v.$$

An element $f \in Z^1(G, M)$ is a *coboundary* if it is of the form $\delta(v)$ for some $v \in M$. We define

$$H^1(G; M) := Z^1(G, M)/\operatorname{Im}\delta.$$

In places (especially Section 4.3), we will assume more familiarity with group cohomology; the reader desiring more background is again referred to [Bro94].

2.2. **The work of Korkmaz and Kasahara.** The present work is deeply indebted to the papers [Kor23, Kas24] of Korkmaz and Kasahara. We recall their main results here.

Theorem 2.1 (Korkmaz, Theorem 1 of [Kor23])**.** *For $g \geqslant 3$ and $n \leqslant 2g-1$, any homomorphism $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ is trivial.*

Theorem 2.2 (Korkmaz, Theorem 2 of [Kor23])**.** *For $g \geqslant 3$, any homomorphism $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(2g, \mathbb{C})$ is either trivial or else is conjugate to the symplectic representation $\Psi : \mathrm{Mod}(S) \to \mathrm{GL}(H)$.*

We now turn to representations of dimension $2g+1$. Let UTS denote the unit tangent bundle of S. When S is non-closed, $UTS \cong S \times S^1$ splits as a product, and in particular, $H_1(UTS; \mathbb{C}) \cong \mathbb{C}^{2g+1}$. Let $\widetilde{H} = H_1(UTS; \mathbb{C})$. Any diffeomorphism of S induces a diffeomorphism of UTS; the *unit tangent representation*

$$\widetilde{\Psi} : \mathrm{Mod}(S) \to \mathrm{GL}(\widetilde{H})$$

is the induced action on homology. Note that $H \cong H_1(S; \mathbb{C})$ arises as a quotient of $\widetilde{H}$ via pushforward along the projection $\pi : UTS \to S$; the induced action on H is evidently Ψ. This representation was studied by Trapp [Tra92]. Kasahara classified representations of $\mathrm{Mod}(S)$ for S a surface of genus g with an arbitrary number of punctures and/or boundary components. In the setting we consider here (where S has at most one boundary component or puncture), his result specializes as follows.

Theorem 2.3 (Kasahara, cf. Theorem 1.1 of [Kas24])**.** *Let S be a surface of genus $g \geqslant 7$ which is either closed, has one boundary component, or one puncture. If S is closed, then every nontrivial representation $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(2g+1, \mathbb{C})$ is conjugate to $H \oplus \mathbb{C}$. If S is nonclosed, then any nontrivial ρ is conjugate to $H \oplus \mathbb{C}$, or to $\widetilde{H}$, or to $\widetilde{H}^*$.*

Remark 2.4. We explain here how to deduce this statement from [Kas24, Theorem 1.1]. The statement there asserts that isomorphism classes of nontrivial representations of dimension $2g+1$ up to dualizing are in bijection with $H^1(\mathrm{Mod}(S); H)/\mathbb{C}^\times$. When S is closed, Morita computed $H^1(\mathrm{Mod}(S); H) = 0$, and when S has one boundary component or puncture, $H^1(\mathrm{Mod}(S); H) \cong \mathbb{C}$ (see [Mor89], recalled here as Proposition 4.3). Thus up to dualizing, there is a unique representation not of the form $H \oplus \mathbb{C}$; it is not hard to show that $\widetilde{H}$ is such a representation. For further discussion of $\widetilde{H}$, see Section 4.1.

Remark 2.5. Our proof depends on Theorems 2.1 and 2.2 of Korkmaz, as well as various technical lemmas established therein. It is logically independent of Kasahara's main theorem (Theorem 2.3), and indeed improves the range of his result from $g \geqslant 7$ down to $g \geqslant 4$, but we again make use of some of the internal technology.

2.3. **The Torelli group and the work of Johnson.** Recall that the *Torelli group* $\mathcal{I}(S) \leqslant \mathrm{Mod}(S)$ is the kernel of the symplectic representation Ψ. The structure of $\mathcal{I}(S)$ was greatly clarified in a series of papers of Dennis Johnson in the 1980's. We recall the relevant portions of his theory here.

We first recall the *Johnson homomorphism*, as developed in [Joh80]. For the sake of expediency we will not describe this in full detail. For our purposes it is sufficient to know that the Johnson homomorphism is a map

$$\tau : \mathcal{I}(S) \to A_S,$$

where A_S is a certain finitely generated torsion-free abelian group. Define the *Johnson kernel*

$$\mathcal{K}(S) = \ker(\tau)$$

as the kernel of τ.

A deep theorem of Johnson shows that $\mathcal{K}(S)$ is generated by just two types of elements. Let $c \subset S$ be a separating curve (i.e. one for which $S \setminus c$ is disconnected). In the case where S has a puncture or boundary component, the *genus* of S is the genus of the subsurface *not* containing this; if S is closed we define the genus as the smaller of the genera of the subsurfaces bounded by c. A *separating twist* is the Dehn twist T_c about a separating curve c; we define the genus of such T_c as the genus of c.

Theorem 2.6 (Johnson, [Joh85])**.** *For $g \geqslant 3$, $\mathcal{K}(S)$ is generated by the set of separating twists of genus one and two.*

The Torelli group is itself generated by elements admitting a simple description. A *bounding pair* is a set of curves $a, b \subset S$ that are disjoint and such that $S \setminus \{a, b\}$ is disconnected. A *bounding pair map* is the product $T_a T_b^{-1}$ of Dehn twists; it is straightforward to show that these are elements of the Torelli group.

Theorem 2.7 (Johnson, [Joh79])**.** *For $g \geqslant 3$, $\mathcal{I}(S)$ is generated by bounding pair maps.*

Though we will not use either of these facts in the present work, it is worth remarking that both $\mathcal{I}(S)$ and $\mathcal{K}(S)$ are in fact generated by a *finite* collection of such elements. This result for $\mathcal{I}(S)$ is due to Johnson [Joh83], while the result for $\mathcal{K}(S)$ is a much more recent result of Ershov-He [EH18].

3. Bi-affine representations and double cocycle suspensions

As discussed in the Introduction, our strategy of proof is to show that every $\mathrm{Mod}(S)$-representation in the dimension range we consider has a special structure, what we call *bi-affine*. The purpose of this section is to establish the basic theory of bi-affine representations. We will see that these can be constructed and classified in terms of group cocycles via a construction known as a *double cocycle suspension*. Throughout this discussion, G is a group and $V \cong \mathbb{C}^m$ is a $\mathbb{C}[G]$-module. Before proceeding further, let us record the basic definitions.

Definition 3.1 (Affine, co-affine, bi-affine)**.** A representation $\rho : G \to \mathrm{GL}(V)$ is *affine* if there is an invariant subspace $W \leqslant V$ for which the quotient representation $\overline{\rho} : G \to \mathrm{GL}(V/W)$ is trivial. ρ is *co-affine* if there is an invariant subspace W for which $\rho|_W$ is trivial. ρ is *bi-affine* if there are invariant subspaces $V_1 \leqslant V_2 \leqslant V$ for which V_1 and V/V_2 are both trivial. In this setting, the quotient V_2/V_1 is called the *core* of the representation.

Remark 3.2. The dual of an affine representation is co-affine, and vice versa.

We first show how to construct (co-)affine representations in terms of cocycles.

Construction 3.3 (Cocycle suspension). Let $\rho : G \to \mathrm{GL}(W)$ be a representation, and let $\phi \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$ be a cocycle. The *cocycle suspension* is the G-module

$$V = W \oplus_\phi \mathbb{C}^b$$

with underlying vector space $W \oplus \mathbb{C}^b$, with $g \in G$ acting via

$$g \cdot (w, u) = (\rho(g)(w) + \phi(g)(u), u).$$

This is manifestly an affine representation of G.

Likewise, given $\rho : G \to \mathrm{GL}(W)$ and $\phi \in Z^1(G, \mathrm{Hom}(W, \mathbb{C}^a))$, one constructs a co-affine representation

$$V = \mathbb{C}^a \oplus_\phi W$$

carrying a G-action via

$$g \cdot (v, w) = (v + \phi(g)(w), \rho(g)(w)).$$

Here, $\mathrm{Hom}(W, \mathbb{C}^a)$ is naturally a *right* G-module, so that

$$\phi(gh) = \phi(h) + \phi(g) \cdot h.$$

Conversely, every (co-)affine representation admits a description as a cocycle suspension.

Lemma 3.4. *Let $\rho : G \to \mathrm{GL}(V)$ be an affine G-representation with invariant subspace W. Then there is a cocycle $\phi \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$ such that there is an isomorphism of G-modules*

$$V \cong W \oplus_\phi \mathbb{C}^b.$$

The analogous statement holds for co-affine representations.

Proof. We consider the case of V affine; the co-affine case follows by dualizing. Let $\mathbb{C}^b \leqslant V$ be a vector space complement to W. Since V is affine, $\rho(g)(u) \equiv u \pmod{W}$ for any $u \in \mathbb{C}^b$ and any $g \in G$. Consequently, there is a function $\phi : G \to \mathrm{Hom}(\mathbb{C}^b, W)$ via the formula

$$\phi(g)(u) = \rho(g)u - u,$$

and this is readily seen to be a cocycle. The direct sum decomposition $V \cong W \oplus \mathbb{C}^b$ then underlies an isomorphism $V \cong W \oplus_\phi \mathbb{C}^b$ of G-modules. □

Up to isomorphism, an affine representation is determined solely by the projective class of the associated cohomology class $[\phi] \in H^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$.

Lemma 3.5. *Let G be a group and let W be a $\mathbb{C}[G]$-module. Let $\phi_1, \phi_2 \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$ be cocycles such that $\lambda[\phi_1] = [\phi_2]$ as elements of $H^1(G; \mathrm{Hom}(\mathbb{C}^b, W))$, for some $\lambda \in \mathbb{C}^\times$. Then*

$$W \oplus_{\phi_1} \mathbb{C}^b \cong W \oplus_{\phi_2} \mathbb{C}^b$$

as $\mathbb{C}[G]$-modules.

Proof. By hypothesis, there is $h_0 \in \mathrm{Hom}(\mathbb{C}^b, W)$ such that $\phi_2 = \lambda\phi_1 + \delta(h_0)$ (recall here the coboundary map $\delta : \mathrm{Hom}(\mathbb{C}^b, W) \to Z^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$, with $\delta(h_0)(g) = (g-1)h_0$). Then one verifies that the map $f : W \oplus_{\phi_1} \mathbb{C}^b \to W \oplus_{\phi_2} \mathbb{C}^b$ given by

$$f(w, u) = (\lambda w + h_0(u), u)$$

yields the desired isomorphism of $\mathbb{C}[G]$-modules. □

The coefficient module $\mathrm{Hom}(\mathbb{C}^b, W)$ naturally carries the structure of a $(G, \mathrm{GL}(b,\mathbb{C}))$-bimodule. Given $\phi \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$ and $B \in \mathrm{GL}(b,\mathbb{C})$, we write ϕ^B for the cocycle

$$\phi^B(g)(v) = \phi(g)(B^{-1}v).$$

The isomorphism type of the affine representation determined by ϕ is likewise unaffected by the $\mathrm{GL}(b,\mathbb{C})$-action.

Lemma 3.6. *With G, W, ϕ, B as above, there is an isomorphism of $\mathbb{C}[G]$-modules*

$$W \oplus_\phi \mathbb{C}^b \cong W \oplus_{\phi^B} \mathbb{C}^b.$$

Proof. The map $f : W \oplus_\phi \mathbb{C}^b \to W \oplus_{\phi^B} \mathbb{C}^b$ given by $f(v,u) = (v, Bu)$ is readily seen to furnish an isomorphism of $\mathbb{C}[G]$-modules. $\square$

Remark 3.7. Dually, given $\phi \in \mathrm{Hom}(W, \mathbb{C}^a)$ and $A \in \mathrm{GL}(a,\mathbb{C})$, define $A\phi \in \mathrm{Hom}(W, \mathbb{C}^a)$ via $(A\phi)(g) = A(\phi(g))$. Then $W \oplus_{A\phi} \mathbb{C}^a \cong W \oplus_\phi \mathbb{C}^a$.

Now let $V_1 \leqslant V_2 \leqslant V$ be the filtration underlying a bi-affine representation. We can view this as an affine representation with invariant subspace V_2, and we can in turn view V_2 as a co-affine representation with invariant subspace V_1. Alternatively, we can view V as a co-affine representation with invariant subspace V_1, and the quotient V/V_1 as an affine representation with invariant subspace V_2/V_1. Thus Lemma 3.4 leads immediately to the following classification of bi-affine representations.

Lemma 3.8. *Let G be a group, and let $V_1 \leqslant V_2 \leqslant V$ be a filtration of vector spaces. Then there is a one-to-one correspondence between bi-affine representations $\rho : G \to \mathrm{GL}(V)$ with core $W = V_2/V_1$, and either of the following sets of data:*

- *Cocycles $\phi_1 \in Z^1(G, \mathrm{Hom}(W, \mathbb{C}^a))$ and $\phi_2^+ \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a \oplus_{\phi_1} W))$, or*
- *Cocycles $\phi_2 \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$ and $\phi_1^+ \in Z^1(G, \mathrm{Hom}(W \oplus_{\phi_2} \mathbb{C}^b, \mathbb{C}^a))$.*

Remark 3.9. We call the constructions $(\mathbb{C}^a \oplus_{\phi_1} W) \oplus_{\phi_2^+} \mathbb{C}^b$ and $\mathbb{C}^a \oplus_{\phi_1^+} (W \oplus_{\phi_2} \mathbb{C}^b)$ *double cocycle suspensions*. We will use the terms "bi-affine" and "double cocycle suspension" essentially interchangeably throughout, preferring the former in situations where it is not necessary to directly discuss the cocycles, and the latter where it is.

A priori, there is a certain unsatisfying asymmetry in the theory, since the same bi-affine V can be described as both an affine and a co-affine representation, and the cocycles obtained via these procedures do not even live in the same spaces. If one thinks in coordinates (specifying the $\mathbb{C}[G]$-module structure as certain block upper-triangular matrices), it is evident that there should be some relationship between these constructions, and that moreover there should be a "symmetric" construction of a double cocycle suspension, based around cocycles $\phi_1 \in Z^1(G, \mathrm{Hom}(W, \mathbb{C}^a))$, $\phi_2 \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$, and some additional data (in coordinates, the data of the top right block, an element of $\mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a)$). In Lemma 3.10 we explain how to achieve this.

Lemma 3.10. *A double cocycle suspension can be constructed from the following data: cocycles $\phi_1 \in Z^1(G, \mathrm{Hom}(W, \mathbb{C}^a))$ and $\phi_2 \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$, and a set function $\alpha : G \to \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a)$ that satisfies the "coboundary equation"*

$$\alpha(g_1 g_2) - \alpha(g_1) - \alpha(g_2) = \phi_1(g_1) \circ \phi_2(g_2). \tag{1}$$

Conversely, given $\phi_1 \in Z^1(G, \mathrm{Hom}(W, \mathbb{C}^a))$ and $\phi_2^+ \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a \oplus_{\phi_1} W))$, define $\alpha : G \to \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a)$ by projecting ϕ_2^+ onto the $\mathbb{C}^a$ factor. Then α satisfies (1). The same is true for $\phi_2 \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, W))$ and $\phi_1^+ \in Z^1(G, \mathrm{Hom}(W \oplus_{\phi_2} \mathbb{C}^b, \mathbb{C}^a)$, defining α by restricting to $\mathbb{C}^b \leqslant H \oplus_{\phi_2} \mathbb{C}^b$.

Proof. There is a short exact sequence of $\mathbb{C}[G]$-modules

$$0 \to \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a) \to \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a \oplus_{\phi_1} W) \to \mathrm{Hom}(\mathbb{C}^b, W) \to 0.$$

The associated long exact sequence in cohomology includes the segment

$$H^1(G; \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a \oplus_{\phi_1} W)) \to H^1(G; \mathrm{Hom}(\mathbb{C}^b, W)) \xrightarrow{\delta} H^2(G; \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a)).$$

This implies that $\phi_2 \in Z^1(G; \mathrm{Hom}(\mathbb{C}^b, W))$ can be lifted to $\phi_2^+ \in Z^1(G; \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a \oplus_{\phi_1} W))$ (and hence give rise to a double cocycle suspension) if and only if $[\delta(\phi_2)] = 0$ in $H^2(G; \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a))$.

An explicit calculation shows that $\delta(\phi_2) \in Z^2(G, \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a))$ is given by the function

$$\delta(\phi_2)(g_1, g_2) = \phi_1(g_1) \circ \phi_2(g_2) \in \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a).$$

Since $\mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a)$ is a trivial $\mathbb{C}[G]$-module, such $\delta(\phi_2)$ is a coboundary if and only if there is $\alpha : G \to \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a)$ such that

$$\alpha(g_1) + \alpha(g_2) - \alpha(g_1 g_2) = \phi_1(g_1) \circ \phi_2(g_2),$$

as claimed.

The converse statements are seen to hold by direct calculation. □

4. Bi-affine representations of the mapping class group

We now specialize to the case of $G = \mathrm{Mod}(S)$. In this section we collect various results about bi-affine representations of the mapping class group. **We assume for the duration of the paper that the core of all bi-affine representations is H, carrying the symplectic representation $\boldsymbol{\Psi}$.** An important preliminary observation is that it suffices to prove Theorem A for the surface $S = \Sigma_{g,1}$ with one boundary component, since the mapping class groups for the punctured and closed surfaces are quotients of this. **Thus, we will assume S is a surface with one boundary component for the remainder of the paper, except where explicitly specified otherwise.**

4.1. The unit tangent representation as an affine representation.

Here we explain how to understand $\widetilde{H}$ and $\widetilde{H}^*$ as a co-affine and affine representation, respectively.

We first recall the construction of $\widetilde{H}$ and its dual $\widetilde{H}^*$. As above, UTS denotes the unit tangent bundle of S, and set $\widetilde{H} = H_1(UTS; \mathbb{C})$. The projection $\pi : UTS \to S$ induces a surjection $\pi_* : \widetilde{H} \to H$. The kernel is the trivial one-dimensional subrepresentation spanned by the class of the S^1 fiber. Thus there is a short exact sequence

$$0 \to \mathbb{C} \to \widetilde{H} \to H \to 0,$$

realizing $\widetilde{H}$ as a co-affine representation. Dually, $\widetilde{H}^* = H^1(UTS; \mathbb{C})$ is seen to be affine, with H embedded as a submodule of codimension one via pullback.

Lemma 4.1. *Let $\phi \in Z^1(\mathrm{Mod}(S); H^*)$ be the cocycle defining $\widetilde{H}$ as a co-affine representation. Then $[\phi] \in H^1(\mathrm{Mod}(S); H)$ is nonzero.*

Proof. See [Mor97, Section 4]. □

4.2. **Extensions.** The next lemma shows that in this setting, any extension of a bi-affine representation is again bi-affine, so long as the dimension does not grow too much.

Lemma 4.2. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a bi-affine representation with respect to the filtration $V_1 \leqslant V_2 \leqslant V$. Suppose that V embeds as a submodule of $\widetilde{V}$, and that $\dim(\widetilde{V}/V_2) = \dim(\widetilde{V}/V) + \dim(V/V_2) \leqslant 2g-1$. Then $\widetilde{V}$ likewise is bi-affine, with respect to the filtration $V_1 \leqslant V_2 \leqslant \widetilde{V}$.*

Dually, suppose $U \leqslant \widetilde{V}$ is a filtration of $\mathbb{C}[G]$-modules, with $V := \widetilde{V}/U$ bi-affine, filtered as $V_1 \leqslant V_2 \leqslant V$. Let $\widetilde{V_1} \leqslant \widetilde{V_2} \leqslant \widetilde{V}$ be the preimage of this filtration in $\widetilde{V}$. Suppose that $\dim(\widetilde{V_1}) = \dim(U) + \dim(V_1) \leqslant 2g-1$. Then $\widetilde{V}$ likewise is bi-affine, with respect to the filtration $\widetilde{V_1} \leqslant \widetilde{V_2} \leqslant \widetilde{V}$.

Proof. By hypothesis, $\dim(\widetilde{V}/V_2) \leqslant 2g-1$. By Theorem 2.1, $\widetilde{V}/V_2$ must be trivial; as V_1 is trivial by hypothesis, this realizes $\widetilde{V}$ as a bi-affine representation. The dual statement follows from this by Remark 3.2. □

4.3. **A cohomological result.** In constructing a double cocycle suspension, one first chooses a cocycle $\phi_1 \in Z^1(G, \mathrm{Hom}(W, \mathbb{C}^a))$, and then chooses a cocycle $\phi_2^+ \in Z^1(G, \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a \oplus_{\phi_1} W))$. Even if $Z^1(G, W)$ is already well-understood, to classify double cocycle suspensions, one needs an understanding of $H^1(G; \mathbb{C}^a \oplus_{\phi_1} W)$ for all cocycles ϕ_1. In Proposition 4.6, we show that this problem has a simple solution in the case of $G = \mathrm{Mod}(S)$ and $W = H$: in the symmetric formulation of Lemma 3.10, *at most one of ϕ_1, ϕ_2 can be cohomologically nontrivial.* This immediately implies Corollary 4.7, which establishes the rather remarkably strong fact that any bi-affine representation of $\mathrm{Mod}(S)$ is the direct sum of a representation of dimension $2g+1$ with a trivial representation.

In this subsection, we will temporarily switch notation, writing $\Sigma_{g,1}$ and $\Sigma_{g,*}$ for the surfaces of genus g with one boundary component (resp. puncture), since the difference between these settings will be live.

We will need to recall a few results about the cohomology of $\mathrm{Mod}(S)$.

Proposition 4.3 (Morita, cf. Propositions 4.1 and 6.4 of [Mor89])**.** *For $S = \Sigma_{g,1}$ or $\Sigma_{g,*}$,*

$$H^1(\mathrm{Mod}(S); H) \cong \mathbb{C}.$$

For $S = \Sigma_g$,

$$H^1(\mathrm{Mod}(S); H) = 0.$$

There are various constructions of crossed homomorphisms representing a generator (over $\mathbb{Z}$) of $H^1(\mathrm{Mod}(S); H)$; see especially the work of Furuta, as recorded by Morita [Mor97, p. 569] and Trapp [Tra92]. While we will not need to delve into the theory, we will need to know one structural property of cocycles in $Z^1(\mathrm{Mod}(S), H)$. We recall the *Birman exact sequence*

$$1 \to \pi_1\Sigma_g \to \mathrm{Mod}_{g,*} \to \mathrm{Mod}_g \to 1,$$

whose kernel $\pi_1\Sigma_g$ is known as the *point-pushing subgroup.*

Lemma 4.4. *Let $\phi \in Z^1(\mathrm{Mod}_{g,*}, H)$ be given. Then the restriction*

$$\phi : \pi_1\Sigma_g \to H$$

to the point-pushing subgroup is given as

$$\phi = \lambda ab$$

for some $\lambda \in \mathbb{C}$, and $ab : \pi_1\Sigma_g \to H_1(\Sigma_{g,*}; \mathbb{Z})$ is the abelianization map. Moreover, $\lambda \neq 0$ if and only if $[\phi] \in H^1(\mathrm{Mod}_{g,*}; H)$ is nonzero.

Proof. The restriction of ϕ to $\pi_1\Sigma_g$ determines a class in $H^1(\pi_1\Sigma_g; H) \cong \mathrm{Hom}(H, H)$ which is invariant under the action of Mod_g, i.e. $\phi : H \to H$ is equivariant with respect to $\mathrm{Sp}(2g, \mathbb{Z})$. As H is an irreducible $\mathrm{Sp}(2g, \mathbb{Z})$-module, by Schur's lemma, $\phi : H \to H$ must be of the form $\phi = \lambda\,\mathrm{id}$ for $\lambda \in \mathbb{C}$.

$\lambda = 0$ if and only if $\phi \in Z^1(\mathrm{Mod}_{g,*}, H)$ is pulled back from $\overline{\phi} \in Z^1(\mathrm{Mod}_g, H)$. According to Proposition 4.3, $H^1(\mathrm{Mod}_g, H) = 0$, so that if $\lambda = 0$ then $[\phi] \in H^1(\mathrm{Mod}_{g,*}; H) = 0$. Conversely, if $\lambda \neq 0$, then the pullback to $H^1(\Sigma_g, H)$ is nontrivial; this proves the second claim. □

We will also require the following technical lemma.

Lemma 4.5. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\pi_1\Sigma_g$-modules. Then the connecting map*

$$\delta : H^1(\pi_1\Sigma_g; C) \to H^2(\pi_1\Sigma_g; A)$$

in the change-of-coefficients long exact sequence admits the following explicit description:

Write $\pi_1\Sigma_g = \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \ldots [x_g, y_g] = 1\rangle$. Given $\phi \in Z^1(\pi_1\Sigma_g, C)$, choose lifts

$$\widetilde{\phi(x_1)}, \widetilde{\phi(y_1)}, \ldots, \widetilde{\phi(x_g)}, \widetilde{\phi(y_g)} \in B$$

of the corresponding elements $\phi(x_i), \phi(y_i) \in C$. Via Poincaré duality, identify $H^2(\pi_1\Sigma_g; A) \cong H_0(\pi_1\Sigma_g; A) \cong A_{\pi_1\Sigma_g}$. Then

$$\delta(\phi) = \sum_{i=1}^{g} \prod_{j=1}^{i-1} [x_j, y_j] \cdot \Big((1 - y_i)\widetilde{\phi(x_i)} + (x_i - 1)\widetilde{\phi(y_i)}\Big)$$

Proof. This can be seen explicitly, by using the standard CW complex for Σ_g (consisting of a single 0- and 2-cell, and $2g$ 1-cells) to obtain a resolution for $\mathbb{Z}[\pi_1\Sigma_g]$ over $\mathbb{Z}$, and computing the connecting map δ using this chain complex. □

Proposition 4.6. *For $S = \Sigma_{g,1}$ or $S = \Sigma_{g,*}$, we have $H^1(\mathrm{Mod}(S), \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a \oplus_\phi H)) \neq 0$ if and only if $[\phi] = 0 \in H^1(\mathrm{Mod}(S); \mathrm{Hom}(H, \mathbb{C}^a))$.*

Proof. If $[\phi] = 0$, then by Lemma 3.5, $\mathbb{C}^a \oplus_\phi H \cong \mathbb{C}^a \oplus H$. Then by Proposition 4.3,

$$H^1(\mathrm{Mod}(S), \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a \oplus_\phi H)) \cong H^1(\mathrm{Mod}(S), \mathbb{C}^{ab} \oplus H^{\oplus b}) \cong \mathbb{C}^b.$$

Conversely, suppose $[\phi] \neq 0$. The $\mathrm{Mod}(S)$-module $\mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^a \oplus_\phi H)$ decomposes as a direct sum of b copies of $\mathbb{C}^a \oplus_\phi H$, and so it suffices to prove the claim under the hypothesis $b = 1$. In this setting, we have $\phi \in Z^1(\mathrm{Mod}(S), \mathrm{Hom}(H, \mathbb{C}^a))$ (recalling from Construction 3.3 that $\mathrm{Hom}(H, \mathbb{C}^a)$ is a *right* $\mathrm{Mod}(S)$-module), giving rise to the left $\mathrm{Mod}(S)$-module $\mathbb{C}^a \oplus_\phi H$, with $g \in \mathrm{Mod}(S)$ acting via

$$g \cdot (u, v) = (u + \phi(g)(v), g \cdot v) \tag{2}$$

(here g acts on $v \in H$ via Ψ as usual). As we are assuming $[\phi] \neq 0$, we seek to show that $H^1(\mathrm{Mod}(S), \mathbb{C}^a \oplus_\phi H) = 0$. Since

$$H^1(\mathrm{Mod}(S), \mathrm{Hom}(H, \mathbb{C}^a)) \cong H^1(\mathrm{Mod}(S), H^*)^{\oplus a}$$

and $[\phi] = 0$ if and only if each of its components in $H^1(\mathrm{Mod}(S), H^*)$ vanishes, it suffices to consider the case $a = 1$.

Here, $\mathbb{C} \oplus_\phi H$ decomposes as a module via the short exact sequence

$$0 \to \mathbb{C} \to \mathbb{C} \oplus_\phi H \to H \to 0.$$

We apply the change-of-coefficients long exact sequence for this sequence of $\mathrm{Mod}(S)$-modules, which includes the portion

$$H^1(\mathrm{Mod}(S);\mathbb{C}) \to H^1(\mathrm{Mod}(S);\mathbb{C}\oplus_\phi H) \to H^1(\mathrm{Mod}(S);H) \xrightarrow{\delta} H^2(\mathrm{Mod}(S);\mathbb{C}).$$

As $g \geqslant 3$, $H^1(\mathrm{Mod}(S);\mathbb{C}) = 0$. In addition, by Proposition 4.3, $H^1(\mathrm{Mod}(S);H) = \mathbb{C}$. Thus, $H^1(\mathrm{Mod}(S);\mathbb{C}\oplus_\phi H) = 0$ if and only if $\delta : H^1(\mathrm{Mod}(S);H) \to H^2(\mathrm{Mod}(S);\mathbb{C})$ is nonzero.

To proceed, we specialize to the case $S = \Sigma_{g,*}$. Here, $H^2(\mathrm{Mod}_{g,*},\mathbb{C}) = \mathbb{C}^2$, spanned by the MMM class e_1 and the *vertical Euler class* e [FM12, Sections 5.5-6]. The vertical Euler class has the property that under the restriction $H^2(\mathrm{Mod}_{g,*},\mathbb{C}) \to H^2(\pi_1\Sigma_g;\mathbb{C})$ induced by the inclusion $\pi_1\Sigma_g \to \mathrm{Mod}_{g,*}$ of the Birman exact sequence, e is sent to a nonzero class in $H^2(\pi_1\Sigma_g;\mathbb{C}) \cong \mathbb{C}$ [FM12, Section 5.5.5].

For an arbitrary $\mathrm{Mod}_{g,*}$-module M, the image of the restriction map $H^k(\mathrm{Mod}_{g,*};M) \to H^k(\pi_1\Sigma_g;M)$ is valued in the invariant submodule $H^k(\pi_1\Sigma_g;M)^{\mathrm{Mod}_g}$. We therefore obtain the following commutative square:

$$\begin{tikzcd}
H^1(\mathrm{Mod}_{g,*};H) \arrow[r, "\delta"] \arrow[d] & H^2(\mathrm{Mod}_{g,*};\mathbb{C}) \arrow[d] \\
H^1(\pi_1\Sigma_g;H)^{\mathrm{Mod}_g} \arrow[r, "\delta"'] & H^2(\pi_1\Sigma_g;\mathbb{C})^{\mathrm{Mod}_g}
\end{tikzcd}$$

It is easy to see that both groups on the bottom are isomorphic to $\mathbb{C}$, with $H^1(\pi_1\Sigma_g;H)^{\mathrm{Mod}_g}$ spanned by the class of the abelianization map

$$ab : \pi_1\Sigma_g \to H.$$

By the discussion of a previous paragraph, the right vertical arrow is a surjection. It follows that if the bottom horizontal arrow is nonzero, then the same holds for the top horizontal arrow.

To compute $\delta : H^1(\pi_1\Sigma_g;H)^{\mathrm{Mod}_g} \to H^2(\pi_1\Sigma_g;\mathbb{C})^{\mathrm{Mod}_g}$, we appeal to Lemma 4.5, as applied to $\phi = ab$ the abelianization map. To that end, lift $\phi(x_i) = [x_i] \in H$ to $(0,[x_i]) \in \mathbb{C}\oplus_\phi H$, and similarly lift $\phi(y_i)$ to $(0,[y_i])$.

By Lemma 4.4, the restriction of $\phi \in Z^1(\mathrm{Mod}_{g,*},H)$ to $\pi_1\Sigma_g$ is of the form $\phi = \lambda ab$ for some $\lambda \in \mathbb{C}$, with $[\phi] \in H^1(\mathrm{Mod}_{g,*};H)$ nonzero if and only if $\lambda \neq 0$. In particular, the restriction of ϕ to $[\pi_1\Sigma_g,\pi_1\Sigma_g]$ is trivial, so that the formula of Lemma 4.5 simplifies as follows:

$$\delta(ab) = \sum_{i=1}^{g} \left((1-y_i)(0,[x_i]) + (x_i - 1)(0,[y_i])\right).$$

From (2), since the action of $\pi_1\Sigma_g \leqslant \mathrm{Mod}_{g,*}$ on H is trivial,

$$(1-y_i)(0,[x_i]) = (-\langle\phi(y_i),x_i\rangle, 0) = -\lambda(\langle y_i,x_i\rangle, 0).$$

Likewise $(x_i-1)(0,[y_i]) = \lambda(\langle x_i,y_i\rangle, 0)$. In total then,

$$\delta(ab) = 2g\lambda,$$

which is nonzero if and only if $\lambda \neq 0$, as was to be shown.

This establishes the result for the case of $S = \Sigma_{g,*}$. For the case of $\Sigma_{g,1}$, we appeal to the Gysin exact sequence for the central extension $0 \to \mathbb{Z} \to \mathrm{Mod}_{g,1} \to \mathrm{Mod}_{g,*} \to 1$, which begins

$$0 \to H^1(\mathrm{Mod}_{g,*};\mathbb{C}\oplus_\phi H) \to H^1(\mathrm{Mod}_{g,1};\mathbb{C}\oplus_\phi H) \to H^0(\mathrm{Mod}_{g,*};\mathbb{C}\oplus_\phi H) \xrightarrow{\smile e} H^2(\mathrm{Mod}_{g,*};\mathbb{C}\oplus_\phi H).$$

From (2), one computes $H^0(\mathrm{Mod}_{g,*};\mathbb{C}\oplus_\phi H) = \mathbb{C}$. As indicated in the diagram, the differential $H^0(\mathrm{Mod}_{g,*};\mathbb{C}\oplus_\phi H) \xrightarrow{\smile e} H^2(\mathrm{Mod}_{g,*};\mathbb{C}\oplus_\phi H)$ is given by taking the cup product with the Euler class of this extension. As this is nonzero (again see [FM12, Section 5.5.5]), exactness implies that there is an isomorphism $H^1(\mathrm{Mod}_{g,*};\mathbb{C}\oplus_\phi H)\cong H^1(\mathrm{Mod}_{g,1};\mathbb{C}\oplus_\phi H)$, proving the result in the remaining case. □

Corollary 4.7. *Let S be a surface of genus $g\geqslant 3$ which is either closed, or has one puncture, or one boundary component. Let V be a bi-affine representation of $\mathrm{Mod}(S)$ of arbitrary dimension. Then either V is trivial, or else $V\cong W\oplus\mathbb{C}^n$, where W is the symplectic representation H, the unit tangent representation $\widetilde{H}$, or the dual $\widetilde{H}^*$. If S is closed, the latter two possibilities cannot arise.*

Proof. Write $V=(\mathbb{C}^a\oplus_{\phi_1}H)\oplus_{\phi_2^+}\mathbb{C}^b$ as a double cocycle suspension, for $\phi_1\in Z^1(\mathrm{Mod}(S);\mathrm{Hom}(H,\mathbb{C}^a))$ and $\phi_2^+\in Z^1(\mathrm{Mod}(S);\mathrm{Hom}(\mathbb{C}^b,\mathbb{C}^a\oplus_{\phi_1}H))$. By Proposition 4.6, if $[\phi_1]\neq 0$ then ϕ_2^+ is a coboundary. If instead $[\phi_1]=0$ in $H^1(\mathrm{Mod}(S);\mathrm{Hom}(H,\mathbb{C}^a))$, then the same is true of $[\phi_1^+]\in H^1(\mathrm{Mod}(S);\mathrm{Hom}(H\oplus_{\phi_2}\mathbb{C}^b,\mathbb{C}^a))$, in light of the exact sequence

$$H^1(\mathrm{Mod}(S);\mathrm{Hom}(\mathbb{C}^b,\mathbb{C}^a))\to H^1(\mathrm{Mod}(S);\mathrm{Hom}(H\oplus_{\phi_2}\mathbb{C}^b,\mathbb{C}^a))\to H^1(\mathrm{Mod}(S);\mathrm{Hom}(H,\mathbb{C}^a))$$

induced by change of coefficients and the vanishing $H^1(\mathrm{Mod}(S);\mathrm{Hom}(\mathbb{C}^b,\mathbb{C}^a))=0$ of the first term (since $\mathrm{Hom}(\mathbb{C}^b,\mathbb{C}^a)$ is a trivial module and $\mathrm{Mod}(S)$ is perfect for $g\geqslant 3$).

Therefore possibly after dualizing, we have that ϕ_2^+ is a coboundary. By Lemma 3.5,

$$V\cong(\mathbb{C}^a\oplus_{\phi_1}H)\oplus\mathbb{C}^b$$

decomposes as the direct sum of an affine representation and a trivial module. Now consider $\phi_1\in Z^1(\mathrm{Mod}(S),\mathrm{Hom}(H,\mathbb{C}^a))$. As in Remark 3.7, the isomorphism type of $\mathbb{C}^a\oplus_{\phi_1}H$ is unaffected by replacing ϕ_1 by $A\phi_1$ for any $A\in\mathrm{GL}(a,\mathbb{C})$. If S is closed, then by Proposition 4.3, necessarily $[\phi_1]=0$, and so $V\cong H\oplus\mathbb{C}^{a+b}$. In the nonclosed case, Proposition 4.3 states that $H^1(\mathrm{Mod}(S);H)\cong\mathbb{C}$, and so the components of $[\phi]$ (viewed as a tuple of a elements of $H^1(\mathrm{Mod}(S);H^*)$) are proportional. Thus there exists $A\in\mathrm{GL}(a,\mathbb{C})$ such that $A\phi_1$ has at most one entry (say, $\phi_{11}\in H^1(\mathrm{Mod}(S);H^*)$) that determines a nontrivial class in $H^1(\mathrm{Mod}(S);H^*)$. Again invoking Lemma 3.5, after a further change of coordinates,

$$\mathbb{C}^a\oplus_{\phi_1}H\cong\mathbb{C}^{a-1}\oplus(\mathbb{C}\oplus_{\phi_{11}}H).$$

If $[\phi_{11}]\in H^1(\mathrm{Mod}(S);H)$ is zero, then $V\cong H\oplus\mathbb{C}^{a+b}$. Otherwise, since $H^1(\mathrm{Mod}(S);\mathbb{C})\cong\mathbb{C}$, by Lemma 3.5 there is a unique isomorphism class of representation of the form $\mathbb{C}\oplus_{\phi_{11}}H$ with $[\phi_{11}]\neq 0$, which by Lemma 4.1 is represented by $\widetilde{H}$. □

In later arguments, we will make use of the following structural corollary of this result.

Corollary 4.8. *Let $\rho:\mathrm{Mod}(S)\to\mathrm{GL}(V)$ be a double cocycle suspension representation on the vector space $V=(\mathbb{C}^a\oplus_{\phi_1}H)\oplus_{\phi_2^+}\mathbb{C}^b$. Let c be a nonseparating simple closed curve, and T_c the associated Dehn twist. Then $\rho(T_c)$ is unipotent and the 1-eigenspace $E_1^c\leqslant V$ is of codimension one.*

Proof. By Corollary 4.7, $V=W\oplus\mathbb{C}^n$, where W is one of $H,\widetilde{H}$, or $\widetilde{H}^*$. It therefore suffices to prove the claim for such $W=H$ and $W=\widetilde{H}^*$. If $W=H$ (and hence $\rho=\Psi$), then $\Psi(T_c)(x)=x+\langle x,c\rangle\,[c]$, which is unipotent and has 1-eigenspace $[c]^\perp\leqslant H$ of codimension one as claimed.

In the case $W=\widetilde{H}^*$, the action of a Dehn twist is given as

$$\widetilde{\Psi}^*(T_c)=\left(\begin{array}{c|c}\Psi(T_c)&\phi(T_c)\\\hline 0&1\end{array}\right),$$

where $\phi:\mathrm{Mod}(S)\to H$ is a crossed homomorphism (the dual of the crossed homomorphism defining $\widetilde{H}$ as in Section 4.1). According to [Kas24, Theorem 4.2], there is a constant $\lambda\in\mathbb{C}$ (depending on ϕ,c, and a choice of orientation of c) such that $\phi(T_c)=\lambda[c]$. As we saw in the previous paragraph, $\Psi(T_c)-I$ likewise has image contained in the span of $[c]$, and annihilates $[c]$, so that $\widetilde{\Psi}^*(T_c)$ has 1-eigenspace of codimension one and is unipotent, as claimed. □

5. A structure theorem

The goal of this section is to prove Proposition 5.4, which gives a sufficient condition under which a representation $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ is bi-affine. We first recall an extremely useful result of Korkmaz which we will use throughout the rest of the paper, as well as two preparatory results about representations of the symplectic group.

Lemma 5.1 (Flag triviality criterion (Korkmaz), Lemma 7.1 of [Kor23]). *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ be a representation. If there is a flag*

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_k = \mathbb{C}^n$$

of $\mathrm{Mod}(S)$-invariant subspaces such that each quotient W_i/W_{i-1} is a trivial $\mathrm{Mod}(S)$-representation, then the image of ρ is trivial. In particular this holds if $\dim(W_i/W_{i-1}) \leqslant 2g-1$ for each $i = 1, 2, \ldots, k$.

Invariant flags turn out to be ubiquitous, in large part because of the following principle. Suppose $R \subset S$ is a subsurface, and let $c \subset S \setminus R$ be a simple closed curve. Then $\mathrm{Mod}(R)$ commutes with T_c, and since commuting linear transformations preserve (generalized) eigenspaces, any flag of generalized eigenspaces for $\rho(T_c)$ is a $\mathrm{Mod}(R)$-invariant flag. We will often use this principle without further comment.

Lemma 5.2. *For $g \geqslant 3$, the irreducible representations of $\mathrm{Sp}(2g, \mathbb{C})$ of smallest dimension are the trivial representation $\mathbb{C}$ of dimension 1, the standard representation H of dimension $2g$, and $\wedge^2 H/\mathbb{C}$ of dimension $\binom{2g}{2} - 1$ (for $g = 3$ there is a second representation of dimension $\binom{2g}{2} - 1 = 14$).*

Proof. The low-dimensional representations of semi-simple Lie groups are tabulated in [AVÈ67, Table 1]. ☐

Recall that a *transvection* on H is a transformation $T_x(y) = y + \langle x, y\rangle\, x$.

Lemma 5.3. *Let V be a $\mathrm{Sp}(2g, \mathbb{Z})$-module of dimension at most $4g-1$. Suppose that every transvection acts unipotently on V. Then V is bi-affine.*

Proof. We claim that $\rho : \mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{GL}(V)$ extends to give a representation of $\mathrm{Sp}(2g, \mathbb{C})$. By Margulis's superrigidity theorem, it is sufficient to show that the Zariski closure $\overline{\mathrm{Im}(\rho)} \leqslant \mathrm{GL}(V)$ is connected (the other hypotheses in the superrigidity theorem being easily observed to hold).

This follows from the unipotence hypothesis. Since each transvection acts unipotently on V, the Zariski closure of the cyclic subgroup it generates is a one-parameter subgroup of the form $\exp(tN)$ for some nilpotent N; in particular, it is connected in the analytic topology. Note that $\overline{\mathrm{Im}(\rho)}$ moreover contains each left and right translate of such subgroup. Since transvections generate $\mathrm{Sp}(2g, \mathbb{Z})$, it follows that one can construct a path in $\overline{\mathrm{Im}(\rho)}$ (in the analytic topology) from the identity to any element of $\mathrm{Im}(\rho)$ by traveling along a succession of cosets of such one-parameter subgroups. Thus $\mathrm{Im}(\rho)$ is contained in an analytically-connected subset of $\overline{\mathrm{Im}(\rho)}$, and hence $\overline{\mathrm{Im}(\rho)}$ is connected in the Zariski topology as well.

Thus ρ extends to a representation $\rho : \mathrm{Sp}(2g, \mathbb{C}) \to GL(V)$. While this need not be semi-simple, V decomposes as a flag $0 \leqslant V_1 \leqslant \ldots \leqslant V_k = V$ of subrepresentations, with each V_{i+1}/V_i irreducible. As $\dim(V) = 4g - 1 < \binom{2g}{2} - 1$, by Lemma 5.2, at most one of these quotients can be nontrivial and any such quotient must be isomorphic to H. If *all* such quotients are trivial, then by Lemma 5.1, the representation of $\mathrm{Sp}(2g, \mathbb{Z})$ (being, *a fortiori*, a representation of $\mathrm{Mod}(S)$), must be trivial.

Otherwise, V decomposes as $0 = V_0 \leqslant V_1 \leqslant V_2 \leqslant V_3 = V$, with all irreducible quotients between V_0 and V_1, and between V_2 and V_3 trivial, and with $V_2/V_1 \cong H$. Again by Lemma 5.1, V_1 and V/V_2 must be trivial, realizing V as bi-affine. ☐

Proposition 5.4. *Let $g \geqslant 3$, and let $S = \Sigma_{g,1}$ be a surface of genus g with one boundary component. Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ be a representation, endowing $V = \mathbb{C}^n$ with the structure of a $\mathrm{Mod}(S)$-module. Suppose that the following conditions hold:*

(A) The restriction of ρ to $\mathcal{K}(S)$ is trivial,
(B) For any nonseparating curve $c \subset S$, $\rho(T_c)$ is unipotent.

Then for $n \leqslant 4g-1$, either V is trivial, or else V is bi-affine.

Proof. We proceed by induction on $m = n - 2g$. Korkmaz's Theorem 2.2 provides the base case $m = 0$. Before proceeding any further, observe that if V satisfies hypotheses (A) and (B), then so does any submodule $W \leqslant V$, and likewise for any quotient module V/W.

We consider the restriction of ρ to the Torelli group $\mathcal{I}(S)$. By hypothesis (A), $\rho|_{\mathcal{I}(S)}$ factors through $\mathcal{I}(S)/\mathcal{K}(S)$. Recalling the discussion of Section 2.3, $\rho|_{\mathcal{I}(S)}$ factors through the Johnson homomorphism $\tau : \mathcal{I}(S) \to A_S$; in particular, the image is abelian. By Theorem 2.7, $\mathcal{I}(S)$ is generated by bounding pair maps $T_a T_b^{-1}$, where a and b are disjoint and homologous. By hypothesis (B), it follows that every bounding pair map acts unipotently. In summary, the action of $\mathcal{I}(S)$ on V is both unipotent and abelian.

In light of this, the fixed space $V^{\mathcal{I}(S)}$ is nonempty. This is a $\mathrm{Mod}(S)$-submodule, and moreover since $\mathcal{I}(S)$ acts trivially, the action of $\mathrm{Mod}(S)$ factors through the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. If $V^{\mathcal{I}(S)} = V$, then the result follows by Lemma 5.3. Otherwise, $V^{\mathcal{I}(S)}$ is a proper submodule of V. By induction, either $V^{\mathcal{I}(S)}$ is bi-affine, or else $V^{\mathcal{I}(S)}$ is trivial.

In the former case, as $\dim(V) \leqslant 4g-1$ and V contains a copy of H of dimension $2g$, the dimension bound of Lemma 4.2 is seen to hold, and so by that result, V is bi-affine. If $V^{\mathcal{I}(S)}$ is trivial, then by Lemma 5.1, the complement $V/V^{\mathcal{I}(S)}$ must be a nontrivial $\mathrm{Mod}(S)$-module. By induction, this must be bi-affine, and again since $\dim(V) \leqslant 4g-1$, the dimension bound of Lemma 4.2 holds and we conclude that V is bi-affine. ∎

6. Proof of Theorem A: unipotence

In these final two sections we undertake the proof of Theorem A. The idea will be to show that both of the conditions of Proposition 5.4 (unipotence and annihilation of the Johnson kernel) are satisfied for all representations of $\mathrm{Mod}(S)$ in dimension at most $3g-3$. This will show that all representations under consideration are bi-affine; Theorem A then follows from Corollary 4.7. In this section we show that unipotence holds, in fact up to dimension $4g-3$.

Proposition 6.1. *Let $g \geqslant 4$, and let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ be a representation. Let $c \subset S$ be a nonseparating simple closed curve. Then for $0 \leqslant n \leqslant 4g-3$, $\rho(T_c)$ is unipotent.*

The proof uses the following lemma, a generalization of [Kor23, Lemma 4.3] with the same proof. Here and throughout, for a simple closed curve $c \subset S$, we write

$$E^c_{\lambda,k} = \ker((\rho(T_c) - \lambda I)^k)$$

for the degree-k generalized λ-eigenspace of $\rho(T_c)$. In the case $k = 1$ of the genuine eigenspace, we will write simply E^c_λ.

Lemma 6.2 (Korkmaz, cf. Lemma 4.3 of [Kor23])**.** *Let $b, c \subset S$ be nonseparating simple closed curves satisfying $i(b, c) = 1$. Let ρ be a linear representation of $\mathrm{Mod}(S)$ and let λ be an eigenvalue of $\rho(T_b)$ (and hence of $\rho(T_c)$ also). If $E^b_{\lambda,k} = E^c_{\lambda,k}$ for some $k \geqslant 1$, then $E^b_{\lambda,k}$ is $\mathrm{Mod}(S)$-invariant.*

We now give the proof of Proposition 6.1:

Proof. We proceed by induction on n. The base cases $n \leqslant 2g$ follow from Korkmaz (Theorem 2.2). For $n > 2g$, suppose for contradiction that $\rho(T_c)$ has an eigenvalue $\lambda \neq 1$. Let $\lambda_\sharp$ denote the multiplicity of λ as an eigenvalue.

Firstly, suppose $\lambda_\sharp = n$. If $\rho(T_c) = \lambda I_n$, applying ϕ to the lantern relation results in $\lambda^4 = \lambda^3$ and hence $\lambda = 1$, a contradiction. Thus

$$1 \leqslant \dim E^c_\lambda \leqslant n - 1 \leqslant 4g - 4.$$

In particular, either E^c_λ or $\mathbb{C}^n/E^c_\lambda$ has dimension between 1 and $2g-2$. Both of these are invariant under $\mathrm{Mod}(S \setminus \{c\})$, where $S \setminus \{c\}$ is a surface of genus $g-1$ obtained by cutting c open in S. Since $g - 1 \geqslant 3$, any representation of $\mathrm{Mod}(S \setminus \{c\})$ with dimension between 1 and $2g-2$ is known by Korkmaz (Theorem 2.2), and in particular Dehn twists act unipotently, so $\lambda = 1$, a contradiction.

Thus $\lambda_\sharp < n$. Let $E^c_{\lambda,gen}$ be the generalized λ-eigenspace. As above, either $E^c_{\lambda,gen}$ or $\mathbb{C}^n/E^c_{\lambda,gen}$ has dimension between 1 and $2g-2$. If $1 \leqslant \dim E^c_{\lambda,gen} \leqslant 2g-2$, by a similar argument as above, we get $\lambda = 1$, a contradiction. Thus $1 \leqslant \dim \mathbb{C}^n/E^c_{\lambda,gen} \leqslant 2g-2$, and $\mathrm{Mod}(S \setminus \{c\})$ acts on this space either trivially or via the symplectic representation; in either case, Dehn twists act unipotently. Therefore any nonseparating simple closed curve on $S \setminus \{c\}$ has the same generalized λ-eigenspace as c. In particular, taking two nonseparating simple closed curves d, d' on $S \setminus \{c\}$ with $i(d, d') = 1$, then by Lemma 6.2, $E^c_{\lambda,gen}$ is $\mathrm{Mod}(S)$-invariant, of dimension $< n$, so by induction, $\lambda = 1$, concluding the proof. $\square$

This also leads to the following corollary:

Corollary 6.3. *Any homomorphism from* $\mathrm{Mod}(S)$ *to a compact Lie group of dimension at most* $4g - 3$ *is trivial.*

Proof. Given $f : \mathrm{Mod}(S) \to G$ such a homomorphism, the adjoint representation furnishes a linear representation of $\mathrm{Mod}(S)$ of dimension $\leqslant 4g - 3$, which sends Dehn twists to unipotent elements by Proposition 6.1. But the only unipotent element in a compact Lie group is the identity matrix [Bor91]. Thus $\mathrm{Im}(f)$ is contained in $Z(G)$, an abelian group, and since $g \geqslant 3$ by standing assumption, it follows that $\mathrm{Im}(f)$ is trivial. $\square$

For the results of the next section, it is also necessary to specify the dimension of the 1-eigenspace. We make use of the following piece of linear algebra that governs the structure of a generalized eigenspace.:

Lemma 6.4 (Jordan inequalities, cf. Section 2.2 of [CS23])**.** *Let* $A \in \mathrm{End}(\mathbb{C}^n)$ *be a linear transformation, and let* λ *be an eigenvalue of* A*. Consider the filtration*

$$0 = E_{\lambda,0} \leqslant E_{\lambda,1} \leqslant E_{\lambda,2} \leqslant \ldots \leqslant E_{\lambda,d} = E_{\lambda,gen}$$

of the generalized eigenspace $E_{\lambda,gen}$*. Then the dimensions of the associated graded quotients form a non-increasing sequence:*

$$\dim(E_{\lambda,j}/E_{\lambda,j-1}) \geqslant \dim(E_{\lambda,j+1}/E_{\lambda,j})$$

for $1 \leqslant j \leqslant d - 1$*.*

Proposition 6.5. *Let* $\rho : \mathrm{Mod}_{g,1} \to \mathrm{GL}(2g + m, \mathbb{C})$ *be a representation with* $0 < m \leqslant g - 3$*. Let* $c \subset S$ *be a nonseparating simple closed curve. Then the* 1*-eigenspace* E^c_1 *of* $\rho(T_c)$ *has codimension at most 1.*

Proof. Having established that T_c acts unipotently on $\mathbb{C}^{2g+m}$, we consider the flag of generalized eigenspaces

$$0 \leqslant E^c_{1,1} \leqslant E^c_{1,2} \leqslant \ldots \leqslant E^c_{1,d} = \mathbb{C}^{2g+m};$$

in this notation, $E^c_{1,1} = E^c_1$ is the genuine eigenspace. This is invariant under the action of $\mathrm{Mod}(S \setminus \{c\})$. By the Jordan inequalities (Lemma 6.4), the sequence of dimensions $\dim(E^c_{1,j}/E^c_{1,j+1})$ is nonincreasing, so that $\dim(E^c_1)$ is an upper bound on the dimension of any such quotient. By the flag triviality criterion (Lemma 5.1), it follows that $\dim(E^c_1) \geqslant 2g-2$.

We now proceed by induction on m, for the hypothesis of ρ being bi-affine. This proposition is a partial step in this induction, however, the total induction will only be accomplished in the next section.

If $\dim(E^c_1) \leqslant 2(g-1)+(m-1)$, then by induction, the restriction of $\rho(\mathrm{Mod}(S \setminus \{c\}))$ to E^c_1 must be bi-affine or else trivial. By Lemma 4.2, it follows that $\rho(\mathrm{Mod}(S \setminus \{c\}))$ itself is either bi-affine or trivial, and then the claim follows from Corollary 4.8.

Otherwise $\mathrm{codim}(E^c_1) \leqslant 2$, so it remains only to rule out the case $\mathrm{codim}(E^c_1) = 2$. Consider a second simple closed curve b with $i(b,c) = 1$. Let R be the surface of genus $g-1$ obtained by removing a tubular neighborhood of b, c. Then by Lemma 6.2, either E^c_1 is invariant under $\rho(\mathrm{Mod}(S))$, or else $E^b_1 \cap E^c_1$ is a $\mathrm{Mod}(R)$-invariant subspace of dimension at most $2(g-1)+(m-1)$. In the former case, applying the inductive hypothesis to the action of $\mathrm{Mod}(S)$ on E^c_1 shows that it is either bi-affine or trivial, and we conclude as in the preceding paragraph. In the latter case, we again conclude by induction that the representation of $\mathrm{Mod}(R)$ on $E^b_1 \cap E^c_1$ is bi-affine or trivial, and the argument finishes along the same lines once again. $\square$

7. Proof of Theorem A: separating twists

In the second half of the argument, we show that representations in our dimension range annihilate the Johnson kernel. Following the results of the previous section, we specialize to the case where Dehn twists act unipotently, with genuine eigenspace of codimension one. We say that a representation $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ is *transvective* if for all nonseparating simple closed curves $c \subset S$, $\rho(T_c)$ is unipotent and $\mathrm{codim}(E^c_1) = 1$.

It turns out that transvective representations have an extremely rigid structure on Dehn twists. In Lemma 7.1-7.5, we establish this theory.

Lemma 7.1. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Then for any nonseparating simple closed curve $c \subset S$, there is $\alpha_c \in V^*$ and $v_c \in V$ for which*

$$\rho(T_c)(x) = x + \alpha_c(x)v_c$$

for all $x \in V$, and $\alpha_c(v_c) = 0$

Proof. Since $\mathrm{codim}(E^c_1) = 1$, there is $\alpha_c \in V^*$ and $v_c \in V$ for which

$$\rho(T_c) - I = \alpha_c v_c.$$

Since $\rho(T_c)$ is unipotent, $\rho(T_c) - I$ is nilpotent, and so necessarily $\alpha_c(v_c) = 0$. $\square$

Lemma 7.2. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Let $a, b \subset S$ be nonseparating simple closed curves. Suppose that either $i(a,b) = 1$, or else that $i(a,b) = 0$ but that a and b are not homologous. Let $\alpha_a, \alpha_b, v_a, v_b$ be as in Lemma 7.1. Then α_a and α_b are linearly independent, and the same is true of v_a and v_b.*

Proof. This is a variant of the method of proof of Lemma 6.2; we recall the argument. Suppose for contradiction that α_a, α_b or v_a, v_b are linearly dependent. For convenience, we will notate the span of α_a by $\mathbb{C}\alpha_a$, and similarly for other objects. Let $c \subset S$ be an additional simple closed curve satisfying the same topological constraints as b (i.e. in the first case, $i(a,c) = 1$ and similarly in the second case). By the change-of-coordinates principle, there is $f \in \mathrm{Mod}(S)$ such that $f(a) = a$ and $f(b) = c$. Then $\rho(f)$ commutes with $\rho(T_a)$ and so preserves $\mathbb{C}\alpha_a$ and $\mathbb{C}v_a$. On the other hand, $\rho(f)$

conjugates $\rho(T_b)$ to $\rho(T_c)$, and so takes $\mathbb{C}\alpha_b$ to $\mathbb{C}\alpha_c$, and $\mathbb{C}v_b$ to $\mathbb{C}v_c$. If $\mathbb{C}\alpha_a = \mathbb{C}\alpha_b$, this shows that likewise $\mathbb{C}\alpha_a = \mathbb{C}\alpha_b$, and similarly for $\mathbb{C}v_a, \mathbb{C}v_b, \mathbb{C}v_c$.

Let $\mathcal{C}$ be the graph with vertices in correspondence with isotopy classes of nonseparating simple closed curves on S, and with edges connecting a, b if $i(a, b) = 1$; let $\mathcal{C}'$ be the graph on the same vertex set, with a joined to b if $i(a, b) = 0$ and a, b are non-homologous. Both $\mathcal{C}$ and $\mathcal{C}'$ are connected for $g \geqslant 3$ (see [FM12, Chapter 4]). Thus if $\mathbb{C}\alpha_a = \mathbb{C}\alpha_b$ for some a, b adjacent in $\mathcal{C}$ or $\mathcal{C}'$, and if c is an arbitrary vertex of $\mathcal{C}^{(')}$, there is a path $c_0 = a, c_1, \ldots, c_k = c$ in $\mathcal{C}^{(')}$. The argument of the above paragraph shows that $\mathbb{C}\alpha_a = \mathbb{C}\alpha_{c_1}$, and successively $\mathbb{C}\alpha_{c_i} = \mathbb{C}\alpha_{c_{i+1}}$, ultimately showing $\mathbb{C}\alpha_a = \mathbb{C}\alpha_c$. The same argument of course works with vectors in place of covectors.

We show that in either situation, this leads to a contradiction. If $\mathbb{C}\alpha_a = \mathbb{C}\alpha_b$ for all pairs of nonseparating simple closed curves on S, then this shows that the codimension-1 subspace $E_1^a = \ker(\alpha_a)$ is ρ-invariant and trivial; by the flag triviality criterion (Lemma 5.1), ρ itself is trivial. If $\mathbb{C}v_a = \mathbb{C}v_b$ for all pairs of curves a, b, we reduce to the previous situation by considering the dual representation. □

Lemma 7.3. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Let $a, b \subset S$ be nonseparating simple closed curves satisfying $i(a, b) = 1$. Let $\alpha_a, \alpha_b, v_a, v_b$ be as in Lemma 7.1. Then*

$$\alpha_a(v_b)\alpha_b(v_a) = -1.$$

Proof. Since $i(a, b) = 1$, the twists T_a, T_b satisfy the braid relation: $T_aT_bT_a = T_bT_aT_b$. A direct computation with the formula of Lemma 7.1 shows that

$$\rho(T_aT_bT_a)(x) = x + (2\alpha_a(x) + \alpha_b(x)\alpha_a(v_b) + \alpha_a(x)\alpha_b(v_a)\alpha_a(v_b))v_a + (\alpha_b(x) + \alpha_a(x)\alpha_b(v_a))v_b$$

and, by reversing the roles of a and b,

$$\rho(T_bT_aT_b)(x) = x + (2\alpha_b(x) + \alpha_a(x)\alpha_b(v_a) + \alpha_b(x)\alpha_a(v_b)\alpha_b(v_a))v_b + (\alpha_a(x) + \alpha_b(x)\alpha_a(v_b))v_a.$$

By Lemma 7.2, v_a and v_b are linearly independent. Comparing coefficients on v_a (and using the fact that α_a is not identically zero, also by Lemma 7.2), we obtain the equation

$$1 + \alpha_a(v_b)\alpha_b(v_a) = 0$$

as desired. □

Lemma 7.4. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Let $a, c \subset S$ be nonseparating simple closed curves that satisfy $i(a, c) = 0$ but are not homologous. Let $\alpha_a, \alpha_c, v_a, v_c$ be as in Lemma 7.1. Then*

$$\alpha_a(v_c) = \alpha_c(v_a) = 0.$$

Proof. We compute

$$\rho(T_cT_a) = x + \alpha_a(x)v_a + (\alpha_c(x) + \alpha_a(x)\alpha_c(v_a))v_c$$

and

$$\rho(T_aT_c) = x + \alpha_c(x)v_c + (\alpha_a(x) + \alpha_c(x)\alpha_a(v_c))v_a.$$

By Lemma 7.2, v_a, v_c are linearly independent. Comparing coefficients on v_a and v_c then yields the desired identities. □

Recall that a *chain* $a_1, \ldots, a_k$ of simple closed curves in S is a set of curves for which $i(a_i, a_{i+1}) = 1$ for $i = 1, \ldots, k-1$ and for which $i(a_i, a_j) = 0$ for $|i - j| > 1$. We say that a chain is *standardly embedded* if the classes $[a_1], \ldots, [a_k]$ are linearly independent in $H_1(S; \mathbb{Z})$.

Lemma 7.5. *Let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be a transvective representation. Let $a_1, \ldots, a_{2k}$ be a standardly-embedded chain; let $v_{a_1}, \ldots, v_{a_{2k}}$ be the vectors associated to $\rho(T_{a_1}), \ldots, \rho(T_{a_{2k}})$ as in Lemma 7.1. Then $[v_{a_1}], \ldots, [v_{a_{2k}}]$ are linearly independent in V, and can be normalized so that $\alpha_{a_i}(v_{a_{i+1}}) = -\alpha_{i+1}(v_{a_i}) = 1$ for $i = 1, \ldots, 2k-1$, and $\alpha_{a_i}(v_j) = 0$ for $|i-j| > 1$.*

Proof. To see that $\{[v_{a_i}]\}$ is linearly independent, suppose

$$w = c_1 v_{a_1} + \cdots + c_{2k} v_{a_{2k}} = 0.$$

Applying T_{a_1} and applying Lemma 7.3 and Lemma 7.4,

$$0 = T_{a_1}(w) = w + \alpha_{a_1}(w) v_a = c_2 \alpha_{a_1}(v_{a_2}) v_a,$$

showing that $c_2 = 0$. Successively applying $T_{a_3}, \ldots, T_{a_{2k-1}}$ then shows that all coefficients $c_{2i} = 0$. Working from the other end, applying $T_{a_{2k}}$ shows that $c_{2k} = 0$; working backwards applying $T_{a_{2k-2}}, \ldots, T_{a_2}$ then shows that all remaining coefficients $c_{2i+1} = 0$ as well.

The desired normalization can be defined by setting

$$c_i = \prod_{j=1}^{i-1} \alpha_{a_j}(v_{j+1})$$

and then defining

$$\alpha'_{a_i} = c_i \alpha_{a_i} \qquad \text{and} \qquad v'_{a_i} = \frac{v_{a_i}}{c_i}.$$

It is then a routine calculation to verify that $\alpha'_i(v'_{a_{i+1}}) = 1$ and the other claimed relations. $\square$

We now specialize to the case of a standardly-embedded chain a, b, c, d of length 4. Following Lemma 7.5, we can extend v_a, v_b, v_c, v_d to a basis of V. In such a basis, after normalizing *à la* Lemma 7.5, the elements $\rho(T_a), \ldots, \rho(T_d)$ have the following matrix expressions:

$$\rho(T_a) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \alpha_a(x) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad \rho(T_b) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \alpha_b(x) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rho(T_c) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \alpha_c(x) \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad \rho(T_d) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & \alpha_d(x) \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Here, it should be understood that the last column records what happens to an arbitrary element lying outside the span of $v_a, \ldots, v_d$.

Proposition 7.6. *Let $g \geqslant 4$, and let $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(2g+m, \mathbb{C})$ be a representation. Then for $0 \leqslant m \leqslant g-3$, the restriction of ρ to $\mathcal{K}(S)$ is trivial.*

Proof. We proceed by induction on m, assuming that every representation of $\mathrm{Mod}(S)$ of dimension $<$ $2g+m$ is bi-affine. By Proposition 6.1 and Proposition 6.5, under this hypothesis, any representation $\rho : \mathrm{Mod}(S) \to \mathrm{GL}(2g+m, \mathbb{C})$ is transvective.

By Theorem 2.6, $\mathcal{K}(S)$ is generated by separating twists of genus 1 and 2, and so it suffices to examine these two mapping classes. Let $a, b, c, d \subset S$ be a standardly-embedded chain. Then a regular neighborhood of $a \cup b$ is a subsurface of genus 1, bounded by a separating curve Δ_1. There is an alternate form of the chain relation [FM12, Section 4.4.1] which asserts

$$T_{\Delta_1} = (T_a^2 T_b)^4.$$

A computation shows that

$$\rho(T_a^2 T_b) = \left(\begin{array}{cccc|c} -1 & 2 & 0 & 0 & 2(\alpha_a(x)+\alpha_b(x)) \\ -1 & 1 & 1 & 0 & \alpha_b(x) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{3}$$

and hence

$$\rho(T_a^2 T_b)^2 = \left(\begin{array}{cccc|c} -1 & 0 & 2 & 0 & 2\alpha_b(x) \\ 0 & -1 & 2 & 0 & -2\alpha_a(x) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right);$$

a final squaring shows that $\rho(T_{\Delta_1}) = \rho(T_a^2 T_b)^4 = I$ as required.

The computation for a separating twist of genus 2 proceeds along the same lines. We use the alternate formulation of the chain relation

$$T_{\Delta_2} = (T_a^2 T_b T_c T_d)^8,$$

where Δ_2 is the boundary of the surface of genus 2 containing a, b, c, d. Picking up from (3),

$$\rho(T_a^2 T_b T_c T_d) = \left(\begin{array}{cccc|c} -1 & 0 & 0 & 2 & 2(\alpha_a(x)+\alpha_b(x)+\alpha_c(x)+\alpha_d(x)) \\ -1 & 0 & 0 & 1 & \alpha_b(x)+\alpha_c(x)+\alpha_d(x) \\ 0 & -1 & 0 & 1 & \alpha_c(x)+\alpha_d(x) \\ 0 & 0 & -1 & 1 & \alpha_d(x) \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Taking successive powers yields

$$\rho(T_a^2 T_b T_c T_d)^2 = \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2\alpha_d(x) \\ 1 & 0 & -1 & -1 & -2\alpha_a(x)-\alpha_b(x)-\alpha_c(x) \\ 1 & 0 & -1 & 0 & -\alpha_b(x)+\alpha_d(x) \\ 0 & 1 & -1 & 0 & -\alpha_c(x)+\alpha_d(x) \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

and

$$\rho(T_a^2 T_b T_c T_d)^4 = \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 2(\alpha_b(x)+\alpha_d(x)) \\ 0 & -1 & 0 & 0 & -2\alpha_a(x) \\ 0 & 0 & -1 & 0 & 2\alpha_d(x) \\ 0 & 0 & 0 & -1 & -2(\alpha_a(x)+\alpha_c(x)) \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

and a final squaring yields $\rho(T_a^2 T_b T_c T_d)^8 = I$. $\square$

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